Mat 2540 HW9

1. How many comparisons are needed for a binary search in a set of 64 elements?

Sol: The divide-and-conquer recurrence relation of a binary search is

$f(n) = f(\frac{n}{2}) + 2$. and $f(1) = 2$

If $n=64$, we have $f(64) = f(32) + 2$

$= f(16) + 2 + 2 = f(16) + 4$

$= f(8) + 2 + 4 = f(8) + 6$

$= f(4) + 2 + 6 = f(4) + 8$

$= f(2) + 2 + 8 = f(2) + 10$

$= f(1) + 2 + 10 = f(1) + 12 = 2 + 12 = 14.$

$\Rightarrow$ Totally it needs 14 comparisons.

7. Suppose that $f(n) = f(n/3) + 1$ when n is a positive integer divisible by 3, and $f(1) = 1$. Find

a) $f(3)$. **b)** $f(27)$. **c)** $f(729)$.

Sol. a) $f(3) = f(\frac{3}{3}) + 1 = f(1) + 1 = 1 + 1 = 2$

b) $f(27) = f(\frac{27}{3}) + 1 = f(9) + 1 = f(\frac{9}{3}) + 1 + 1 = \underbrace{f(3)}_{\text{from (a)}} + 2 = 2 + 2 = 4$

c) $f(729) = f(\frac{729}{3}) + 1 = f(243) + 1 = f(\frac{243}{3}) + 1 + 1 = f(81) + 2$

$= f(\frac{81}{3}) + 1 + 2 = \underbrace{f(27)}_{\text{from (b)}} + 3 = 4 + 3 = 7$

9. Suppose that $f(n) = f(n/5) + 3n^2$ when n is a positive integer divisible by 5, and $f(1) = 4$. Find

a) $f(5)$. **b)** $f(125)$. **c)** $f(3125)$.

Sol: a) $f(5) = f(\frac{5}{5}) + 3 \cdot 5^2 = f(1) + 75 = 4 + 75 = 79$

b) $f(125) = f(\frac{125}{5}) + 3 \cdot 125^2 = f(25) + 3 \cdot 125^2 = f(\frac{25}{5}) + 3 \cdot 25^2 + 3 \cdot 125^2$

$= f(5) + 3 \cdot 25^2 + 3 \cdot 125^2 = 79 + 3 \cdot 25^2 + 3 \cdot 125^2$

c) $f(3125) = f(\frac{3125}{5}) + 3 \cdot 3125^2 = f(625) + 3 \cdot (3125)^2$

$= f(\frac{625}{5}) + 3 \cdot (625)^2 + 3 \cdot (3125)^2 = f(125) + 3 \cdot (625)^2 + 3 \cdot (3125)^2$

$= 79 + 3 \cdot 25^2 + 3 \cdot 125^2 + 3 \cdot 625^2 + 3 \cdot 3125^2$

10. Find $f(n)$ when $n = 2^k$, where f satisfies the recurrence relation $f(n) = f(n/2) + 1$ with $f(1) = 1$.

Sol: $f(2^k) = f(\frac{2^k}{2}) + 1 = f(2^{k-1}) + 1 = f(\frac{2^{k-1}}{2}) + 1 + 1 = f(2^{k-2}) + 2$

$= f(\frac{2^{k-2}}{2}) + 1 + 2 = f(2^{k-3}) + 3 = \cdots$

$= f(2^{k-(k-1)}) + k - 1 = f(2^1) + k - 1 = f(\frac{2}{2}) + 1 + k - 1$

$= f(1) + k = 1 + k$

11. Give a big-O estimate for the function f in Exercise 10 if f is an increasing function.

Sol: If $n = 2^k$, then $k = \log_2 n$. since $f(2^k) = 1 + k$, we have

$f(2^k) = 1 + \log_2 n \Rightarrow f(2^k)$ is $O(\log_2 n)$ or $O(\log n)$

12. Find $f(n)$ when $n = 3^k$, where f satisfies the recurrence relation $f(n) = 2f(n/3) + 4$ with $f(1) = 1$.

Sol: Using the theorem in 6. from Classwork 12, we have

$\alpha = 2$, $c = 4$

If $n = 3^k$, we have $\beta = 3$ and

$f(3^k) = 2^k f(1) + 4 \cdot \frac{2^k - 1}{2 - 1} = 2^k + 4(2^k - 1) = 5 \cdot 2^k - 4$

13. Give a big-O estimate for the function f in Exercise 12 if f is an increasing function.

Sol: If $n = 3^k$, then $k = \log_3 n$. since $f(3^k) = 5 \cdot 2^k - 4$, then we have

$f(3^k) = 5 \cdot 2^{\log_3 n} - 4 = 5 \cdot n^{\log_3 2} - 4$ which is $O(n^{\log_3 2})$

14. Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

Sol: Let f be a recurrence relation for the number of rounds in the elimination tournament. Then we have

$f(n) = f(\frac{n}{2}) + 1$, $f(2) = 1$

15. How many rounds are in the elimination tournament described in Exercise 14 when there are 32 teams?

$f(n) = f(\frac{n}{2}) + 1$, $f(2) = 1$.

$f(32) = f(16) + 1 = f(8) + 1 + 1 = f(4) + 1 + 1 + 1 = f(2) + 1 + 1 + 1 + 1 = 5.$

16. Solve the recurrence relation for the number of rounds in the tournament described in Exercise 14.

Sol: $f(n) = f(\frac{n}{2}) + 1$, $f(2) = 2$.

Let $n = 2^k$, we have $f(2^k) = f(\frac{2^k}{2}) + 1 = f(2^{k-1}) + 1$

$= f(2^{k-2}) + 2 = f(2^{k-3}) + 1 + 2 = f(2^{k-3}) + 3$

$= \cdots = f(2^{k-(k-1)}) + k - 1 = f(2) + k - 1$

$= 1 + (k-1) = k$.

17. Suppose that the votes of n people for different candidates (where there can be more than two candidates) for a particular office are the elements of a sequence. A person wins the election if this person receives a majority of the votes.

a) Devise a divide-and-conquer algorithm that determines whether a candidate received a majority and, if so, determine who this candidate is. [*Hint:* Assume that n is even and split the sequence of votes into two sequences, each with $n/2$ elements. Note that a candidate could not have received a majority of votes without receiving a majority of votes in at least one of the two halves.]

b) Use the master theorem to give a big-O estimate for the number of comparisons needed by the algorithm you devised in part (a).

Sol: a) Basis step

21. Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

a) Find $f(16)$.

b) Give a big-O estimate for $f(n)$. [*Hint:* Make the substitution $m = \log n$.]

Sol (a) $f(16) = 2f(\sqrt{16}) + 1$

$= 2f(4) + 1 = 2(2f(\sqrt{4}) + 1) + 1$

$= 4f(2) + 3 = 4 \cdot 1 + 3 = 7$

(b)