

Mat 2540 HW7

1. Use mathematical induction to verify the formula derived in Example 2 for the number of moves required to complete the Tower of Hanoi puzzle.

Proof: Given $H_n = 2H_{n-1} + 1$. To prove $H_n = 2^n - 1$ by induction.
with $H_1 = 1$.

We have

Basis step: $1 = H_1 = 2^1 - 1 = 1$ (checked)

Inductive step: Assume $H_k = 2^k - 1$ for all integer $k > 0$.

We have $H_{k+1} = 2H_k + 1$

$$= 2 \cdot (2^k - 1) + 1$$

$$= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

By mathematical induction, $H_n = 2^n - 1$ for all integer $n > 0$.

3. A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Sol: a) Let a_n be the number of ways to deposit n dollars.

There are 3 possible cases for a_n :

Case 1 The last deposited cash is 1-dollar coin:

The number of ways before the last is a_{n-1}

Case 2 The last deposited cash is \$1 bills.

The number of ways before the last is a_{n-1}

Case 3 The last deposited cash is \$5 bills.

The number of ways before the last is a_{n-5}

$$\Rightarrow a_n = a_{n-5} + 2a_{n-1}, \quad a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 16$$

b) $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 16$

$$c) a_8 = a_3 + 2a_7 = 8 + 2 \cdot 140 = 288$$

$$a_9 = a_4 + 2a_8 = 16 + 2 \cdot 288 = 16 + 576 = 592$$

$$a_{10} = a_5 + 2a_9 = 33 + 2 \cdot 592 = 33 + 1184 = 1217$$

4. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

Sol: Let a_n be the number of ways to pay a bill of n pesos

There are 9 possible cases:

Case 1 The last deposited cash is 1 peso coin
The number of ways before the last is a_{n-1}

Case 2 The last deposited cash is 2 pesos coin
The number of ways before the last is a_{n-2}

Case 3 The last deposited cash is 5 pesos coin
The number of ways before the last is a_{n-5}

Case 4 The last deposited cash is 10 pesos coin
The number of ways before the last is a_{n-10}

Case 5 The last deposited cash is 5 pesos bill
The number of ways before the last is a_{n-5}

Case 6 The last deposited cash is 10 pesos bill
The number of ways before the last is a_{n-10}

Case 7 The last deposited cash is 20 pesos bill
The number of ways before the last is a_{n-20}

Case 8 The last deposited cash is 50 pesos bill
The number of ways before the last is a_{n-50}

Case 9 The last deposited cash is 100 pesos bill
The number of ways before the last is a_{n-100}

$$\Rightarrow a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}.$$

7. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
 b) What are the initial conditions?
 c) How many bit strings of length seven contain two consecutive 0s?

Sol a) Let a_n be the number of bit strings of length n that contains a pair of consecutive 0s.

Case 1 end with 1: $\overbrace{\text{length } n-1 \text{ containing a pair of consecutive 0s}}^{n-1 \text{ bits}} 1 \Rightarrow a_{n-1} \text{ ways}$

Case 2 end with 0: $\overbrace{\text{length } n-2 \text{ containing a pair of consecutive 0s}}^{n-2 \text{ bits}} 10 \Rightarrow a_{n-2} \text{ ways}$

$\overbrace{\text{1 or 0 for } n-2 \text{ spots}}^{n-2} .00 \Rightarrow 2^{n-2} \text{ ways}$

$$\Rightarrow a_n = a_{n-1} + a_{n-2} + 2^{n-2}$$

b) $a_1 \Rightarrow$ Does NOT Exist

$a_2 = 1$ (which is $\{00\}$)

$a_3 = 3$ (which are $\{100, 001, 000\}$)

$a_4 = 3 + 1 + 2^2 = 8$ (which are $\{1100, 1001, 0011, 1000, 0100, 0010, 0001, 0000\}$)

c) $a_5 = 8 + 3 + 2^3 = 19$

$a_7 = 19 + 8 + 2^5 = 94$

$a_6 = 19 + 8 + 2^4 = 43$

initial conditions

8. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.
 b) What are the initial conditions?
 c) How many bit strings of length seven contain three consecutive 0s?

Sol Let a_n be the number of bit strings of length n containing 3 consecutive 0s.

For a_n , there are 3 cases:

a) Case 1 end with 1: $\overbrace{\text{length of } n-1 \text{ containing 3 consecutive 0s}}^{\text{length of } n-1} 1 : a_{n-1} \text{ ways}$

Case 2 end with 0: $\overbrace{\text{length of } n-2 \text{ containing 3 consecutive 0s}}^{\text{length of } n-2} 10 : a_{n-2} \text{ ways}$

$\overbrace{\text{length of } n-3 \text{ containing 3 consecutive 0s}}^{\text{length of } n-3} 00 : a_{n-3} \text{ ways}$

$\overbrace{\text{1 or 0 for each spot}}^{\text{length of } n-3} 000 : 2^{n-3}$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$$

b) $a_1, a_2 \Rightarrow$ Does NOT Exist.

$a_3 = 1$ (which is $\{000\}$)

$a_4 = 3$ (which are $\{1000, 0001, 0000\}$)

$a_5 = 8$ (which are $\{11000, 10001, 00011, 10000, 01000, 00010, 00001, 00000\}$)

} initial conditions

c) $a_6 = 1+3+8+2^3 = 20$

$a_7 = 3+8+20+2^4 = 47$

9. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

Sol: Let a_n be the number of bit strings of length n that do not contain 3 consecutive 0s.

a) For a_n , we have 3 cases:

Case 1 end with 1: $\frac{\text{length of } n-1 \text{ bits containing no 3 consecutive 0s}}{n-1 \text{ bits}}$ a_{n-1} ways

Case 2 end with 0: $\frac{\text{length of } n-2 \text{ bits containing no 3 consecutive 0s}}{n-2 \text{ bits}}$ a_{n-2} ways

$\frac{\text{length of } n-3 \text{ bits containing no 3 consecutive 0s}}{n-3 \text{ bits}}$ a_{n-3} ways

$a_n = a_{n-1} + a_{n-2} + a_{n-3}, n > 3$

b) $a_1 = 2$ (which are $\{1, 0\}$)

$a_2 = 4$ (which are $\{11, 10, 01, 00\}$)

$a_3 = 7$ (which are $\{111, 110, 101, 011, 100, 010, 001\}$)

$a_4 = 13$ (which are $\{1111, 1110, 1101, 1011, 0111, 1100, 1001, 0011, 1010, 0101, 0100, 0010, 0110\}$)

} initial conditions

c) $a_5 = 13+7+4 = 24$

$a_6 = 24+13+7 = 44$

$a_7 = 44+24+13 = 81$

11. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.
 b) What are the initial conditions?
 c) In how many ways can this person climb a flight of eight stairs?

Sol: Let a_n be the number of ways to climb n stairs for taking one stair or two stairs at a time.

a) For a_n , we have two cases:

Case 1 the climbing ends with one stair:
 there are $n-1$ stairs before that which has a_{n-1} ways

Case 2 the climbing ends with 2 stairs:
 there are $n-2$ stairs before that which has a_{n-2} ways

Then $a_n = a_{n-1} + a_{n-2}$, $n > 2$

b) *initial condition* $\left\{ \begin{array}{l} a_1 = 1 \text{ (only 1 way when there is only 1 stair)} \\ a_2 = 2 \text{ (climbing 2 stairs: 2 ways (two 1-stair or one 2-stair))} \\ a_3 = 3 \text{ (three 1-stair, one 1-stair then one 2-stair, or one 2-stair then one 1-stair)} \end{array} \right.$

c) $a_4 = a_3 + a_2 = 5$, $a_5 = 5 + 3 = 8$, $a_6 = 8 + 5 = 13$, $a_7 = 13 + 8 = 21$
 $a_8 = 21 + 13 = 34$

13. a) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s.
 b) What are the initial conditions?
 c) How many ternary strings of length six do not contain two consecutive 0s?

Sol: Let a_n be the number of ternary strings of length n containing no 2 consecutive 0s.

a) For a_n , we have 3 cases

Case 1 The strings end with 2 $\underbrace{\text{length of } n-1 \text{ bits containing no 2 consecutive 0s}}_{\geq}$: a_{n-1} ways

Case 2 The strings end with 1 $\underbrace{\text{length of } n-1 \text{ bits containing no 2 consecutive 0s}}_{\text{I}}$ $\underbrace{\text{I}}_{n-1 \text{ bits}}$: a_{n-1} ways

Case 3 The strings end with 0 $\underbrace{\text{length of } n-2 \text{ bits containing no 2 consecutive 0s}}_{\text{I}}$ $\underbrace{\text{I 0}}_{n-2 \text{ bits}}$: a_{n-2} ways
 $\underbrace{\text{I 0 0}}_{n-2 \text{ bits}}$: a_{n-2} ways

Then $a_n = 2a_{n-1} + 2a_{n-2}$, $a > 2$.

b) $a_1 = 3$ (which are $\{0, 1, 2\}$)
 initial conditions $a_2 = 8$ (which are $\{01, 10, 02, 20, 11, 12, 21, 22\}$)
 $a_3 = 2a_1 + 2a_2$ (which are $\{112, 102, 012, 222, 202, 022, 212, 122, 111, 101, 011, 221, 201, 021, 211, 121, 110, 210, 010, 020, 120, 220\}$)
 $= 6 + 6 = 12$

c) $a_4 = 2 \cdot 8 + 2 \cdot 12 = 40$

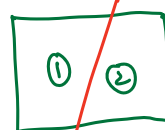
$a_5 = 2 \cdot 12 + 2 \cdot 40 = 104$

$a_6 = 2 \cdot 40 + 2 \cdot 104 = 248$

21. a) Find the recurrence relation satisfied by R_n , where R_n is the number of regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.
 b) Find R_n using iteration.

Sol: Let R_n be the number of regions that a plane is divided into by n lines.

a) $R_0 = 1$ (no line with 1 plane), $R_1 = 2$

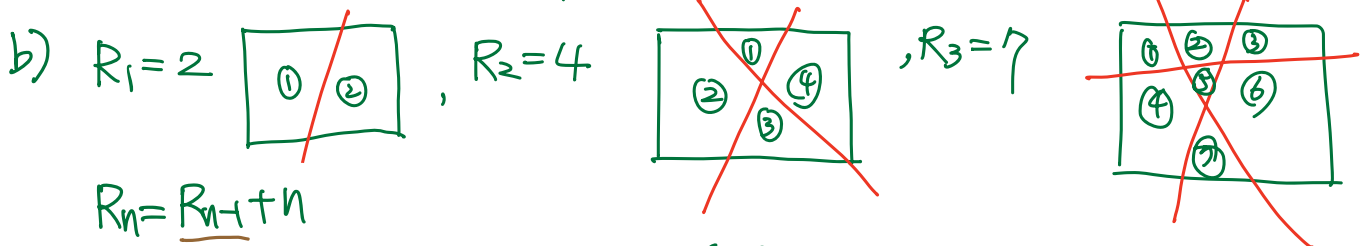


From R_{n-1} to R_n , we add n -th line on a plane with $n-1$ lines

To get the most of it, the n -th line will intersect all the existed $n-1$ lines with $n-1$ intersection points.

Since these $n-1$ intersection points split the n -th line into n segments and each segments will split each region from R_{n-1} into 2 regions. Then there are totally n new regions which are created from adding the n -th line.

Therefore $R_n = R_{n-1} + n \quad n > 1$



$R_n = R_{n-1} + n$

$= (R_{n-2} + n - 1) + n = R_{n-2} + (n - 1) + n$

$= (R_{n-3} + n - 2) + (n - 1) + n = R_{n-3} + (n - 2) + (n - 1) + n$

$= \dots = R_1 + 2 + 3 + \dots + n = R_0 + 1 + 2 + 3 + \dots + n = R_0 + \frac{n(n+1)}{2}$

$= 1 + \frac{n(n+1)}{2}$

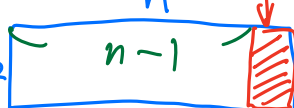
$\Rightarrow R_n = 1 + \frac{n(n+1)}{2}$

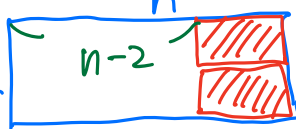
26. a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

b) What are the initial conditions for the recurrence relation in part (a)?
 c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?







Sol: Let a_n be the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

(a) For a_n , there are 2 cases:

Case 1 The final step is put a 1×2 vertically:  There are a_{n-1} ways before this final step.

Case 2 The final step is put two 1×2 horizontally:  There are a_{n-2} ways before this final step

Therefore, $a_n = a_{n-1} + a_{n-2}$, $n > 2$

Initial conditions (b) $a_1 = 1$ (only 1 way: )
 $a_2 = 2$ (There are 2 ways:  )
 $a_3 = a_1 + a_2 = 1 + 2 = 3$ (There are 3 ways:   )

(c) $a_4 = 3 + 2 = 5$, $a_5 = 5 + 3 = 8$, $a_6 = 8 + 5 = 13$, $a_7 = 13 + 8 = 21$
 $a_8 = 21 + 13 = 34$, $a_9 = 34 + 21 = 55$, $a_{10} = 55 + 34 = 89$,
 $a_{11} = 89 + 55 = 144$, $a_{12} = 144 + 89 = 233$, $a_{13} = 233 + 144 = 377$
 $a_{14} = 377 + 233 = 610$, $a_{15} = 610 + 377 = 987$, $a_{16} = 987 + 610 = 1597$
 $a_{17} = 1597 + 987 = 2584$

27. a) Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.

b) What are the initial conditions for the recurrence relation in part (a)?
 c) How many ways are there to lay out a path of seven tiles as described in part (a)?

Sol: Let A_n be the number of ways to lay out a walkway with n tiles which are red, green or gray tiles without two red tiles being adjacent.

(a) For A_n , we have 2 cases:

Case 1 There is one tile away from finishing the walkway: (end with non-red tile)
 $n-1$ tiles without 2 red ones adjacent $\left\{ \begin{array}{l} \text{green} \\ \text{gray} \end{array} \right.$: A_{n-1} ways

$n-1$ tiles without 2 red ones adjacent $\left\{ \begin{array}{l} \text{green} \\ \text{gray} \end{array} \right.$: A_{n-1} ways

Case 2 There are two tiles away from finishing the walkway: (end with red tile)
 $n-2$ tiles without 2 red ones adjacent $\left\{ \begin{array}{l} \text{green red} \\ \text{gray red} \end{array} \right.$: A_{n-2} ways

$n-2$ tiles without 2 red ones adjacent $\left\{ \begin{array}{l} \text{green red} \\ \text{gray red} \end{array} \right.$: A_{n-2} ways

$$\Rightarrow A_n = 2A_{n-1} + 2A_{n-2} \quad n > 2$$

(b) $A_1 = 3$ (either green, gray, or red)
 $A_2 = 8$ (There are 8 ways: (r, green), (r, gray), (green, green), (green, r), (gray, r), (gray, gray), (gray, green), (green, gray))

c) $A_3 = 2A_1 + 2A_2 = 22$, $A_4 = 2 \cdot 8 + 2 \cdot 22 = 60$, $A_5 = 2 \cdot 22 + 2 \cdot 60 = 164$

$$A_6 = 2 \cdot 60 + 2 \cdot 164 = 448, \quad A_7 = 2 \cdot 164 + 2 \cdot 448 = 1224.$$

28. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, \dots$, together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \dots$.

Sol. Given Fibonacci numbers be $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0, f_1 = 1$. Then we have

$$\begin{aligned} f_n &= f_{n-2} + f_{n-3} + f_{n-2} = 2f_{n-2} + f_{n-3} \\ &= 2(f_{n-3} + f_{n-4}) + f_{n-3} = 3f_{n-3} + 2f_{n-4} \\ &= 3(f_{n-4} + f_{n-5}) + 2f_{n-4} = 5f_{n-4} + 3f_{n-5} \end{aligned}$$

with $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3$.

For f_{5n} , we have

$$f_{5n} = 5f_{5n-4} + 3f_{5n-5}$$

To show f_{5n} is divisible by 5, we show it by induction:

Basis Step: ^{when} $n=1$, we have $f_{5 \cdot 1} = 5f_{5 \cdot 1 - 4} + 3f_{5 \cdot 1 - 5} = 5f_1 + 3f_0 = 5 \cdot 1 + 3 \cdot 0 = 5$ which is divisible by 5.

Inductive step: Assume when $n=k$, f_{5k} is divisible by 5. We may say that $f_{5k} = 5 \cdot c$ where c is a constant.

Then, when $n=k+1$, we have

$$\begin{aligned} f_{5(k+1)} &= 5 \cdot f_{5(k+1)-4} + 3f_{5(k+1)-5} \\ &= 5 \cdot f_{5k+1} + 3f_{5k} \end{aligned}$$

Since f_{5k} is divisible by 5, so is $5f_{5k+1}$.

then $f_{5(k+1)}$ is also divisible by 5.

By induction, f_{5n} is divisible by 5 for $n=1, 2, 3, \dots$.