Mat 2540 HW6

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

Sol: Since there are five weekdays (boxes), then, for any set of six classes (objects), there must be at least two classes (objects) on a certain weekday (box) by Pigeonhole principle

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?

Sol: since all the socks are unmatched, then there are 24 socks with half brown and half black.

(a) Since there are two different colors (boxes), then, in order to get a pair of same color, you need take at least 3 shocks (objects) by pigeonhole principle

(b) In order to get at least 2 black socks, this couldn't use pigeonhole principle and the worst scenario is that you take 14 socks which are 12 browns plus 2 black.

7. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Sol: Let a_1, a_2, a_3, a_4, a_5 be any five integers and r_1, r_2, r_3, r_4, r_5 be their remainder when divided by 4, respectively. Since, for the remainder when divided by 4, there are only 4 different possibilities (boxes): 0, 1, 2, 3, then, for r_1, r_2, r_3, r_4, r_5, there must be at least two of them are the same by pigeonhole principle.

9. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Sol: Let $a, a+1, a+2, \ldots, a+n-1$ be a set of n consecutive integers.

Case 1. if a is divisible by n ($a \bmod n \equiv 0$), we're done. for any $a>0$.

Case 2. if a is not divisible by n, assume $a \bmod n \equiv r$ where $0 < r \leq n-1$. Then we have $n-r$ which

$$1 \leq n-r < n$$

and
$$a+(n-r) \bmod n = a \bmod n + (n-r) \bmod n$$
$$= r + (n-r) \equiv 0$$

and $a+(n-r)$ is an integer between a and $a+n-1$.

By Case 1 & 2, there is exactly one integer divisible by n for any n consecuctive integer.

11. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Sol: Assume there are N students. By the general pigeonhole principle, there are at least $\lceil \frac{N}{50} \rceil$ come from the same state.

Therefore, $\lceil \frac{N}{50} \rceil \geqslant 100 \Rightarrow \frac{N}{50} > 100-1 = 99$

$\Rightarrow N > 50 \cdot 99 = 4950 \Rightarrow N$ is at least 4951

17. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

Sol: In order to add up to 7 for 2 numbers from $\{1,2,3,4,5,6\}$, the possible subsets are $\{1,6\}$, $\{2,5\}$, $\{3,4\}$. (3 boxes)

By Pigeonhole principle, at least 4 numbers must be selected from $\{1,2,3,4,5,6\}$ such that there would be a pair of numbers who add up to 7.

21. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

(boxes)

Sol: (a) For three possible categories (freshman / sophomore / junior), there is at least $\lceil \frac{25}{3} \rceil$ students in one of the categories

and $\lceil \frac{25}{3} \rceil = \lceil 8\frac{1}{3} \rceil = 9$ students

(b) Assume the assumption (at least 3 freshmen, or at least 19 sophomore, or at lease five junior) was false,

It means there were only

at most 2 freshmen

at most 18 sophomores, and

at most 4 juniors, and

the total numbe of students are $2+18+4=24$.

However, there are 25 students which is a contradiction.

Therefore, the assumption is incorrect and the statement "at least 3 freshmen, or at least 9 sophomores, or at least 5 juniors" is true.