Mat 2540 HW5

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Sol (a) $\underbrace{C_1^{18}}_{\text{1 math}} \cdot \underbrace{C_1^{325}}_{\text{1 cs}} = 18 \cdot 325 = 5850$

(b) total $18 + 325 = 343$ students $\Rightarrow$ pick 1 student : $C_1^{343} = \frac{343!}{342!\,1!} = 343$

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Sol:

(a) $\overbrace{4 \cdot 4 \cdot 4 \cdot \cdots \cdot 4}^{10 \text{ questions}} = 4^{10}$

each question has 4 options

(b) each question has 4 options plus blank $\Rightarrow$ 5 options for each

Therefore, there are 5^{10} ways that one can answer the questions

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

Sol

$\text{NY} \xrightarrow[\text{6 flights}]{} \text{Denver} \xrightarrow[\text{7 flights}]{} \text{San Franciso}$

Total is $6 \times 7 = 42$ different pairs

7. How many different three-letter initials can people have?

Sol: Different three-letter initials satisfy ① order matters ② letters can be repeated.

Thus, there are 26^3 ways.

9. How many different three-letter initials are there that begin with an *A*?

Sol: Three-letter initials with fixed first letter (which is A) means only two spots to be filled and there are 26^2 ways.

11. How many bit strings of length ten both begin and end with a 1?

Sol. 10-bit strings with fixed begin and end (which are 1s) means only 8-bit spots to be filled and there are 2 options (0 or 1) can be put into these spots. Thus, there are $2^8 = 256$ ways

13. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

n strings.

15. How many strings are there of lowercase letters of length four or less, not counting the empty string?

Sol:

There are $\underset{\text{length of one}}{26} + \underset{\text{length of 2}}{26^2} + \underset{\text{length of 3}}{26^3} + \underset{\text{length of 4}}{26^4} = 475254$

19. How many 6-element RNA sequences

a) do not contain U?
b) end with GU?
c) start with C?
d) contain only A or U?

Sol: For RNA sequence, there are four bases: A, U, C, G

(a) For a 6-element RNA sequences without "U", there are only 3 options (A, C, G) for 6 spots and we have $3^6 = 729$ ways

(b) For a 6-element RNA sequences end with GU, there are only 4 spots left to be filled and we have $4^4 = 256$ ways

(c) For a 6-element RNA sequences start with C, there are only 5 spots left to be filled and we have $4^5 = 1024$ ways

(d) For a 6-element RNA sequence only with A or U, we have $2^6 = 64$ ways to fill.

21. How many positive integers between 50 and 100

a) are divisible by 7? Which integers are these?
b) are divisible by 11? Which integers are these?
c) are divisible by both 7 and 11? Which integers are these?

Sol: The number of the integers between 50 and 100 which is divided by n is $\lfloor \frac{100}{n} \rfloor - \lfloor \frac{50}{n} \rfloor$

a) $n=7$. $\lfloor \frac{100}{7} \rfloor - \lfloor \frac{50}{7} \rfloor = 14 - 7 = 7$ positive numbers and they are

56, 63, 70, 77, 84, 91, 98

b) $n=11$ $\lfloor \frac{100}{11} \rfloor - \lfloor \frac{50}{11} \rfloor = 9 - 4 = 5$ positive numbers and they are

55, 66, 77, 88, 99

c) If a number is divided by both 7 and 11, then this number is divided by LCM(7, 11) = 77. Thus, $\lfloor \frac{100}{77} \rfloor - \lfloor \frac{50}{77} \rfloor = 1 - 0 = 1$ and it is 77.

25. How many strings of three decimal digits

a) do not contain the same digit three times?

b) begin with an odd digit?

c) have exactly two digits that are 4s?

Sol:

a) #(No same digit three times for 3 decimal digits)

#(All possible ways) − #(same digit 3 times)

$= 10 \times 10 \times 10 - 10 = 990$

b) #(strings begin with an odd digit)

$= 5 \times 10 \times 10 = 500$

(only 5 odd numbers: 1, 3, 5, 7, 9)

c) #(strings have exactly two 4s)

$= 9 \times C_1^3 = 9 \times 3 = 27$

except 4 there are 9 options

pick a spot out of 3 to put this non-4 digit

33. How many strings of eight English letters are there

a) that contain no vowels, if letters can be repeated?

b) that contain no vowels, if letters cannot be repeated?

c) that start with a vowel, if letters can be repeated?

d) that start with a vowel, if letters cannot be repeated?

e) that contain at least one vowel, if letters can be repeated?

f) that contain exactly one vowel, if letters can be repeated?

g) that start with X and contain at least one vowel, if letters can be repeated?

h) that start and end with X and contain at least one vowel, if letters can be repeated?

Sol:

There are 5 vowels and 21 consonants.

(a) #(no vowels, letters can be repeated)

$= 21^8$

b) #(no vowels, letters can't be repeated) $= P(21, 8) = \frac{21!}{15!}$

c) #(start with a vowel, letter can be repeated) $= 5 \times 26^7$

d) #(start with a vowel, letter can't be repeated) $= 5 \times P(25, 7) = 5 \cdot \frac{25!}{18!}$

e) #(at least one vowel, letter can be repeated)

$= \#(\text{all ways}) - \#(\text{no vowel})$

$= 26^8 - 21^8$

f) $\#(\text{exactly one vowel, letter can be repeated}) = \underset{\text{choose one vowel}}{5} \cdot \underset{\text{the other 7 are consonants}}{21^7} \times \underset{\text{put the selected vowel in a spot}}{8}$

g) $\#(\text{start w/ X, contain at least 1 vowel, letter can be repeated})$

$= \#(\text{start w/ X, can be repeated}) - \#(\text{start w/ X, can be repeated, no vowel})$

$= 26^7 - 21^7$

h) $\#(\text{start and end with X, contain at least 1 vowel, letter can be repeated})$

$= \#(\text{start and end w/ X, can be repeated}) - \#(\text{start, end w/ X, can be repeated, no vowel})$

$= 26^6 - 21^6$

37. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

a) that are one-to-one?

b) that assign 0 to both 1 and n?

c) that assign 1 to exactly one of the positive integers less than n?

Sol (a) 2 functions when $n=2$: $\{1,2\} \to \{0,1\}$

(b) Besides giving 0 to both 1 & n, there are other $n-2$ spots and each spot has two options (0 or 1), then there are 2^{n-2} functions for $n>1$

(c) $\#(\text{assign 1 to an positive integer less than } n) \cdot \#(n \text{ can be assigned 1 or 0})$

$= (n-1) \cdot 2 = 2(n-1)$

47. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

$$\frac{6!}{6 \cdot 2} = \frac{5!}{2} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot 1}{\cancel{2}} = 60$$

with same neighbors

if left and right are the same

49. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

a) the bride must be next to the groom?
b) the bride is not next to the groom?
c) the bride is positioned somewhere to the left of the groom?

Sol a) If we bond bride & groom together, there are 2 ways (b,g) or (g,b).
Besides bride/groom group, there are the other 4 people, therefore, there are $2 \cdot 5! = 2 \cdot 120 = 240$ ways

b) If the bride is not next to groom, it means there are always people between them.
First, we place the four people in a row, then there are five spots we can choose to put bride or groom in.
So the total is $\underbrace{4!}_{\text{4 ppl in a row}} \cdot \underbrace{P(5,2)}_{\text{5 spots picked 2 for groom and bride}} = 24 \cdot \frac{5!}{3!} = 24 \cdot 5 \cdot 4 = 480$

c) We first place four people in a row, there are 5 spots to put groom and bride when
1) groom and bride stand next to each other with (b, g)
2) groom and bride don't stand next to each other.
We have: $4! \cdot (C_1^5 + C_2^5) = 24(5+10) = 24 \times 15 = 360$

51. How many bit strings of length 10 either begin with three 0s or end with two 0s?

#(10-bit string begin with three 0s) $= 2^7$
#(10-bit string end with two 0s) $= 2^8$
#(10-bit string both begin with three 0s and end with two 0s) $= 2^5$
Total is $2^7 + 2^8 - 2^5 = 128 + 256 - 32 = 352$