MAT2540, Classwork16, Spring2026

10.3 Representing Graphs and Graph Isomorphism Graphs (Conti.)

7. Observations of the Adjacency Matrix For a $\boxed{\text{Directed}}$ Graph

Loops: If there is a loop, it shows on the diagonal of the matrix as "1"

Sum of the row/column: sums of rows represent out-degree, sums of columns represent in-degree

Symmetric property: since lower triangle of the matrix is not mirroring the upper triangle of the matrix.

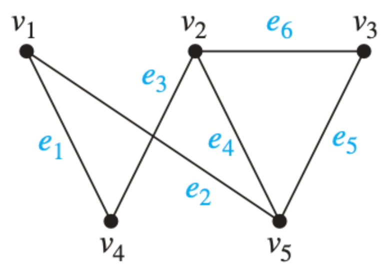
NO, it is NOT symmetric

8. *Definition:* **Incidence Matrix** based on the Incidence of Vertices and Edges of an **Undirected Graph**.

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \dots, v_n$ are the vertices and $e_1, e_2, \dots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix is $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise} \end{cases}.$$

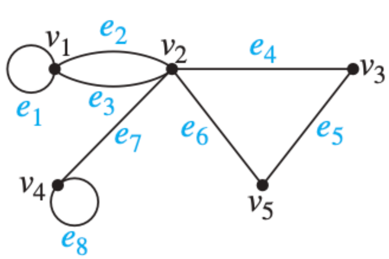
9. Example. Represent the given graph with an incidence matrix of an undirected simple graph.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 2 \\ 2 \\ 3 \end{array}$$

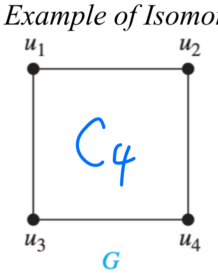
sum of row = deg (of vertex)

10. Example. Represent the given graph with an incidence matrix of a pseudograph.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

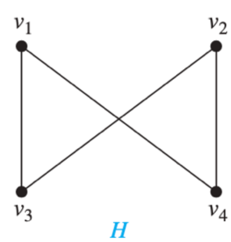
11. *Example of Isomorphism*. Find the $\boxed{\text{Adjacency Matrix}}$ of each given graph and the degrees of each vertex.



G

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

vertex	u_1	u_2	u_3	u_4
degree	2	2	2	2



H

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

vertex	v_1	v_2	v_3	v_4
degree	2	2	2	2

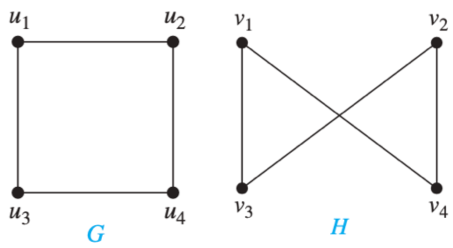
Are these two graphs "the same"?

12. *Definition:* **Isomorphism** of Graphs.

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. Such a function f is called isomorphism.

Two simple graphs that are not isomorphic are called non isomorphic.

13. Find the **isomorphism** f from definition in 12 to show that the graphs $G = (V, E)$, $H = (W, F)$ are isomorphic.



$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ v_1 \\ \boldsymbol{v_4} \\ v_3 \\ \boldsymbol{v_2} \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \boldsymbol{v_4} & v_3 & \boldsymbol{v_2} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The function f with $f(u_1)=v_1$, $f(u_2)=v_4$, $f(u_3)=v_3$, $f(u_4)=v_2$ is a one-to-one and onto function, which is an isomorphism

14. *Definition:* **Graph Invariant**

A property preserved by isomorphism of graphs is called a ____________________. The isomorphic simple graphs need

(1) _______________________: Because there is a one-to-one correspondence between the sets of vertices of the graphs.

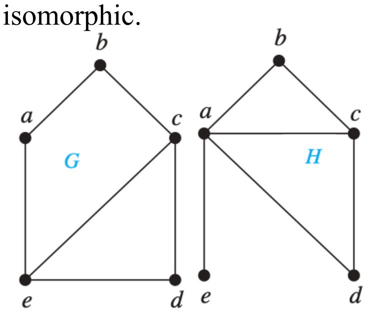
(2) _______________________: Because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges.

(3) The __________________ of the vertices in isomorphic simple graphs must be _______________.

If (1), (2), and (3) are satisfied, then check if ____.

It might be hard to prove two graphs are isomorphic, but it is easier to show two graphs are not isomorphic.

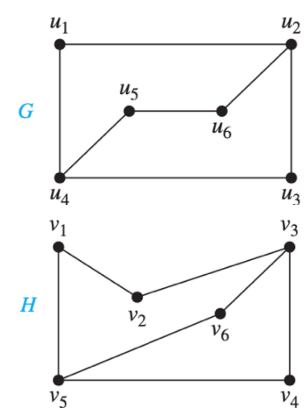
15. *Example.* Check if an **isomorphism** f exists for the given graphs $G = (V, E)$, $H = (W, F)$.



The Graph	The number of vertices	The number of edges	degree sequence
G	$\lvert V\rvert =$	$\lvert E\rvert =$	
H	$\lvert W\rvert =$	$\lvert F\rvert =$	

Conclusion:

16. *Example.* Check if an **isomorphism** f exists for the given graphs $G = (V, E)$, $H = (W, F)$.

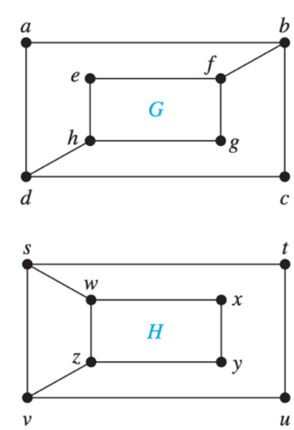


The Graph	The number of vertices	The number of edges	degree sequence
G	$\lvert V\rvert =$	$\lvert E\rvert =$	
H	$\lvert W\rvert =$	$\lvert F\rvert =$	

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} \; & \; & \; & \; & \; & \; \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} \; & \; & \; & \; & \; & \; \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

Conclusion:

17. *Example.* Check if an **isomorphism** f exists for the given graphs $G = (V, E)$, $H = (W, F)$.



The Graph	The number of vertices	The number of edges	degree sequence
G	$\lvert V\rvert =$	$\lvert E\rvert =$	
H	$\lvert W\rvert =$	$\lvert F\rvert =$	

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} \; & \; & \; & \; & \; & \; & \; & \; \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ s \\ t \\ u \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccccccc} s & t & u & v & w & x & y & z \end{array} \\ \begin{bmatrix} \; & \; & \; & \; & \; & \; & \; & \; \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array}$$

Conclusion: