MAT2540, Classwork15, Spring2026

10.2 Graph Terminology and Special Types of Graphs (Conti.)

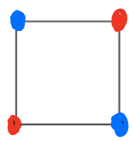
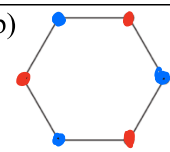
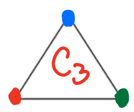
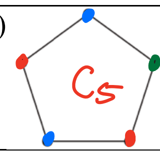
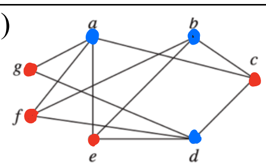
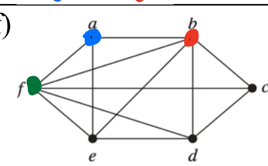
13. Definition: bipartite

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

14. *Theorem*. If one uses the same color to draw any non-adjacent vertices from a graph, Then

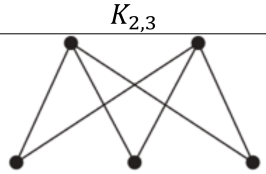
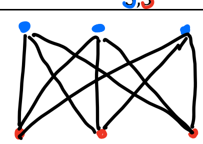
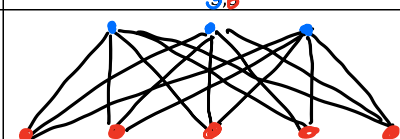
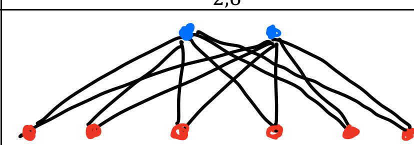
a graph is bipartite if and only if it is 2-colorable.

15. *Examples*. Check if each given graph is a bipartite or not.

(a)  Yes, it is a bipartite	(b)  Yes, it is a bipartite
(c)  NO, it is NOT a bipartite	(d)  NO, it is not a bipartite
(e)  Yes, it is a bipartite	(f)  NO, it is not a bipartite

16. *Theorem*. An undirected graph is bipartite if and only if it does not contain an odd cycle. C_n, n is odd number

17. *Complete Bipartite Graphs*. A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

$K_{2,3}$	$K_{3,3}$	$K_{3,5}$	$K_{2,6}$
			

18. Theorem. An undirected graph has an even number of vertices of odd degree. (why?)

3, 3, 3, 2, 2, 2
sum of degree has to be even

19. A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

Determine whether each of these sequences is graphic.

(a) 5, 4, 3, 2, 1, 0 (b) 2, 2, 2, 2, 2, 2 (c) 1, 1, 1, 1, 1, 1

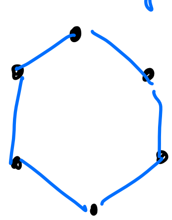
(a) $5+4+3+2+1+0 = 15$

$\frac{15}{2}$ is NOT an integer

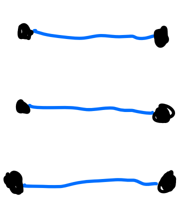
NOT a graph

(b) $\frac{2+2+2+2+2+2}{2} = 6$

Yes, it is a graph



(c) $\frac{1+1+1+1+1+1}{2} = 3$



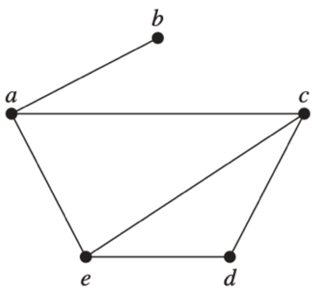
10.3 Representing Graphs and Graph Isomorphism

1. *Definition:* Adjacency Matrix based on the Adjacency of Vertices of an **Undirected Graph**.

Suppose that $G = (V, E)$ is an **undirected graph** where $|V| = n$. Suppose that $v_1, v_2, \dots, v_n$ are the vertices of G. The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero–one matrix with __m__ as its (i, j)th entry when __v_i and v_j are adjacent__ with __multiplicity m__, and __0__ as its (i, j)th entry when __they are adjacent__. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} m & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Representation of Graph:* Adjacency List and Adjacency Matrix For a Simple Graph.



Vertex	Adjacent Vertices	deg.
a	b, c, e	3
b	a	1
c	a, d, e	3
d	c, e	2
e	a, c, d	3

column

upper

row

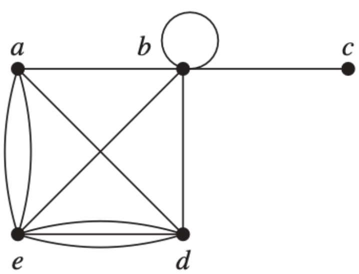
lower

$$\begin{array}{c|ccccc|c} & a & b & c & d & e & \\ \hline a & 0 & 1 & 1 & 0 & 1 & 3 \\ b & 1 & 0 & 0 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 & 1 & 3 \\ d & 0 & 0 & 1 & 0 & 1 & 2 \\ e & 1 & 0 & 1 & 1 & 0 & 3 \\ \hline & 3 & 1 & 3 & 2 & 3 & \end{array}$$

diagonal

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

3. *Representation of Graph:* Adjacency List and Adjacency Matrix For a Multigraph.



Vertex	Adjacent Vertices	deg.
a	b, d, e	4
b	a, b, c, d, e	6
c	b	1
d	a, b, e	5
e	a, b, d	6

sum of row

$$\begin{array}{c|ccccc|c} & a & b & c & d & e & \\ \hline a & 0 & 1 & 0 & 1 & 2 & 4 \\ b & 1 & 2 & 1 & 1 & 1 & 6 \\ c & 0 & 1 & 0 & 0 & 0 & 1 \\ d & 1 & 1 & 0 & 0 & 3 & 5 \\ e & 2 & 1 & 0 & 3 & 0 & 6 \\ \hline & 4 & 6 & 1 & 5 & 6 & \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 3 & 0 \end{bmatrix}$$

4. Observations of the Adjacency Matrix For an Undirected Graph

Loops: If there is a loop, it shows on the diagonal of matrix as "2"

Sum of the row/column: Both sums of row and sums of column represent the degree of individual vertex

Symmetric property: The lower triangle of the matrix is mirroring the upper triangle of the matrix

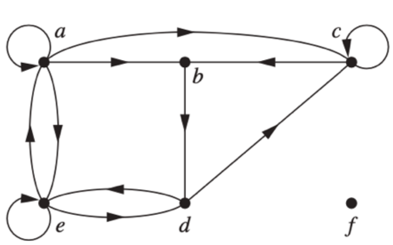
Yes, it's symmetric.

5. *Definition:* Adjacency Matrix based on the Adjacency of Vertices of a **Directed Graph**.

Suppose that $G = (V, E)$ is a **directed graph** where $|V| = n$. Suppose that $v_1, v_2, \dots, v_n$ are the vertices of G. The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero–one matrix with __m__ as its (i, j)th entry when __there is a directed edge from v_i to v_j with multiplicity m__, and __zero otherwise__.
In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} m & \text{if } (v_i, v_j) \text{ is a directed edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

6. *Representation of Graph:* Adjacency Matrix For a Directed Graph.



	In-deg.	Out-deg.
a	2	4
b	2	1
c	3	2
d	2	2
e	3	3
f	0	0

sum of row

$$\begin{array}{c|cccccc|c} & a & b & c & d & e & f & \\ \hline a & 1 & 1 & 1 & 0 & 1 & 0 & 4 \\ b & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ d & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ e & 1 & 0 & 0 & 1 & 1 & 0 & 3 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

sum of column 2 2 3 2 3 0

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$