MAT2540, Classwork14, Spring2026

10.2 Graph Terminology and Special Types of Graphs

1. *Definition*: Fundamental

Two vertices u and v in an **undirected graph** G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incidents with the vertices u and v and e is said to connect u and v.

2. *Definition*: Neighborhood

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.
If A is a subset of V, we denote by $N(A)$, the set of all vertices in G that are adjacent to at least one vertex in A
So,

$$N(A) = \bigcup_{v \in A} N(v)$$

3. *Definition*: Degree

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.
Isolated: A vertex of degree zero is called isolated.
Pendant: A vertex is pendant if and only if it has degree one.

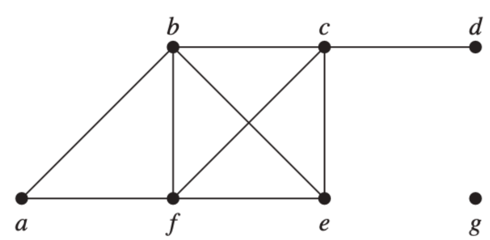
4. *Theorem* (the handshaking theorem). Let $G = (V, E)$, be an undirected graph with m edges. Then

$|E| = m$

number of edge

$$2m = \sum_{v \in V} \deg(v)$$

5. *Example*. What are the degrees and what are the neighborhoods of the vertices in the graph?

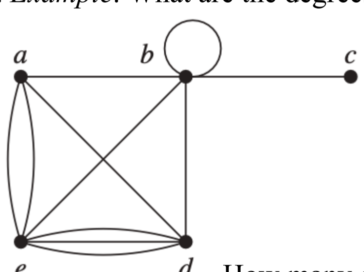


Vertex	Degree	neighborhoods
a	$\deg(a)=2$	$N(a)=\{b,f\}$
b	4	$N(b)=\{a,c,f,e\}$
c	4	$\{b,d,f,e\}$
d	1	$\{c\}$
e	3	$\{b,c,f\}$
f	4	$\{a,b,c,e\}$
g	0	$N(g)=\emptyset$

$N(\{a, c, d\}) = \{b, c, d, f, e\}$

How many edges do the graph have? 9. What is sum of the degree of all the vertices? 18.

6. *Example*. What are the degrees and what are the neighborhoods of the vertices in the graph?



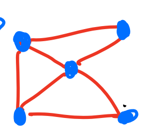
Vertex	Degree	neighborhoods
a	4	$N(a)=\{b,d,e\}$
b	$4+2=6$	$N(b)=\{a,b,c,d,e\}$
c	1	$N(c)=\{b\}$
d	5	$N(d)=\{a,b,e\}$
e	6	$N(e)=\{a,b,d\}$

How many edges do the graph have? 11. What is sum of the degree of all the vertices? 22 ✓

Note that the theorem in 4. applies even if multiple edges and loops are present

7. The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

$$|E| = \frac{4+3+3+2+2}{2} = 7$$



~~4, 3, 3, 2, 1~~ $\Rightarrow$ NOT a graph

8. *Definition*:

When (u, v) is an edge of the graph G with **directed edges**, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u, v), and v is called the end or terminal vertex of (u, v). The initial vertex and terminal vertex of a loop are the same.

9. *Definition*: Degree of vertex of directed graph

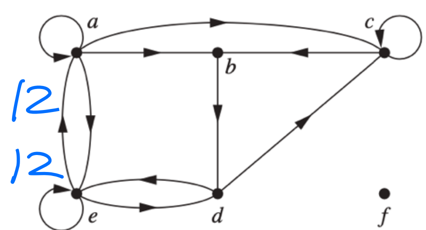
In a graph with directed edges the in-degree (taker) of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree (giver) of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

10. *Theorem*: Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|$$

11. *Example*. Find the in-degree and out-degree of each vertex in the given graph.

Vertex	a	b	c	d	e	f	
In-degree (taker)	2	2	3	2	3	0	12
Out-degree (giver)	4	1	2	2	3	0	12



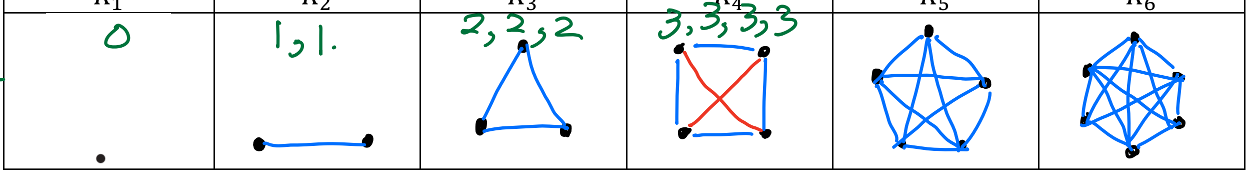
Also check if this graph satisfies the theorem in ~~9~~ 10.

$\Sigma \deg^-(v) = 12$, $\Sigma \deg^+(v) = 12$ $|E| = 12$

12. Special simple graphs.

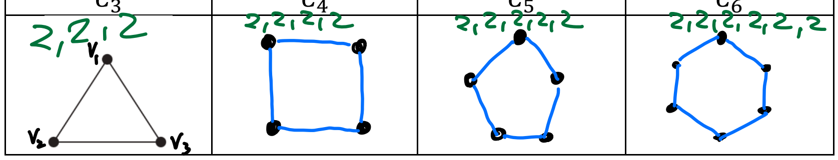
(1) *Complete Graph*. A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

	K_1	K_2	K_3	K_4	K_5	K_6
degree sequence	0	1, 1.	2, 2, 2	3, 3, 3, 3	4, 4, 4, 4, 4	5, 5, 5, 5, 5, 5



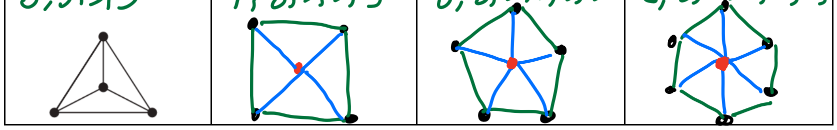
(2) *Cycles*. A cycle $C_n, n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

	C_3	C_4	C_5	C_6
degree sequence	2, 2, 2	2, 2, 2, 2	2, 2, 2, 2, 2	2, 2, 2, 2, 2, 2



(3) *Wheels*. A wheel W_n is adding an additional vertex to a C_n, for , $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

	W_3	W_4	W_5	W_6
degree sequence	3, 3, 3, 3	4, 3, 3, 3, 3	5, 3, 3, 3, 3, 3	6, 3, 3, 3, 3, 3, 3



(4) *n-Cubes*. An n -dimensional hypercube, or n-cube, denoted by Q_n, is a graph that has vertices representing the $2n$ bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

Q_1	Q_2	Q_3