MAT2540, Classwork10, Spring2026

8.2 Solving Linear Recurrence Relations (Conti.)

5. Theorem 1: The Linear Homogeneous of Degree Two Case with Distinct Roots.

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \text{ if and only if } a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, \ldots,$$

where α_1 and α_2 are constants.

$a_n = r^n$, $r^n = c_1 r^{n-1} + c_2 r^{n-2}$ $\to$ $r^n - c_1 r^{n-1} - c_2 r^{n-2} = 0$ $\Rightarrow r^{n-2}(r^2 - c_1 r - c_2) = 0$

$r^{n-2} \cdot r^2 - c_1 r^{n-2} \cdot r - c_2 r^{n-2} = 0$ $\quad r^{n-2} \neq 0$ $\Rightarrow r^2 - c_1 r - c_2 = 0$

6. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $\boxed{a_0 = 2 \text{ and } a_1 = 7}$?

Let $a_n = r^n$, $r^n = r^{n-1} + 2 \cdot r^{n-2} \Rightarrow r^n - r^{n-1} - 2r^{n-2} = 0$

$\Rightarrow r^{n-2}(r^2 - r - 2) = 0 \overset{r \neq 0}{\Longrightarrow} r^2 - r - 2 = 0 \Rightarrow (r+1) \cdot (r-2) = 0 \Rightarrow r = -1 \text{ or } r = 2$

$r \quad +1$
$r \quad -2$

$a_n = \alpha_1 (-1)^n + \alpha_2 (2)^n$

$2 = a_0 = \alpha_1 (-1)^0 + \alpha_2 (2)^0$
$7 = a_1 = \alpha_1 (-1)^1 + \alpha_2 (2)^1$
$\Rightarrow$
$2 = \alpha_1 + \alpha_2$ — (1)
$7 = -\alpha_1 + 2\alpha_2$ — (2)

$(1) + (2) \Rightarrow 9 = 3\alpha_2 \Rightarrow \alpha_2 = 3$

$2 = \alpha_1 + 3 \Rightarrow \alpha_1 = -1$

$\Rightarrow a_n = -1 \cdot (-1)^n + 3(2)^n$

7. Find an explicit formula for the Fibonacci numbers. $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$

Let $f_n = r^n$, $f_{n-1} = r^{n-1}$, $f_{n-2} = r^{n-2}$ $\to$ $r^n = r^{n-1} + r^{n-2} \Rightarrow r^n - r^{n-1} - r^{n-2} = 0 \Rightarrow r^{n-2}(r^2 - r - 1) = 0$

$\overset{r \neq 0}{\Longrightarrow} r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

$0 = f_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0$
$1 = f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1$

$\to$ $0 = \alpha_1 + \alpha_2 \Rightarrow 0 = \left(\frac{-1+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{-1+\sqrt{5}}{2}\right)\alpha_2$

$1 = \left(\frac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_2$

$\Rightarrow 1 = \sqrt{5}\,\alpha_1 \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}} \Rightarrow \alpha_2 = -\frac{1}{\sqrt{5}}$

$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$

8. Theorem 2: The Linear Homogeneous of Degree Two Case with Repeated Root.

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \text{ if and only if } a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n \text{ for } n = 0, 1, 2, \ldots,$$

where α_1 and α_2 are constants.

9. The Multiplicity of A Root.

If a root r_0 is repeated, its multiplicity is the number of time a factor $(r - r_0)$ appears in the factorization

10. What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $\boxed{a_0 = 1 \text{ and } a_1 = 6}$? $a_2 = 36 - 9 = 27$

Let $a_n = r^n$, $a_{n-1} = r^{n-1}$, $a_{n-2} = r^{n-2}$ $\to$ $r^n = 6r^{n-1} - 9r^{n-2} \Rightarrow r^n - 6r^{n-1} + 9r^{n-2} = 0$

$\overset{r \neq 0}{\Longrightarrow} r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$

$r \quad -3$
$r \quad -3$

$a_n = \alpha_1 (3)^n + \alpha_2 n (3)^n$

$1 = a_0 = \alpha_1 (3)^0 + \alpha_2 0 (3)^0$
$6 = a_1 = \alpha_1 (3)^1 + \alpha_2 1 (3)^1$
$\Rightarrow$
$1 = \alpha_1$
$6 = 3\alpha_1 + 3\alpha_2$
$\Rightarrow \alpha_2 = 1$

$\Rightarrow a_n = (3)^n + n(3)^n$

$a_2 = 9 + 18 = 27$

11. Find the solution of the recurrence relation $h_n = 2h_{n-1}$ with $h_1 = 1$.

$r \neq 0$

Let $h_n = r^n$, $r^n = 2r^{n-1} \Rightarrow r^n - 2r^{n-1} = 0 \Rightarrow r^{n-1}(r-2) = 0 \Rightarrow r-2=0 \Rightarrow r=2$

$h_n = \alpha_1 (2)^n$, since $1 = h_1 = \alpha_1 (2)^1 \Rightarrow \alpha_1 = \frac{1}{2} \Rightarrow h_n = \frac{1}{2}(2)^n$ or 2^{n-1}

12. From 1., we know the explicit formula for H_n where $H_n = 2H_{n-1} + 1$ with $H_1 = 1$ is $H_n = 2^n - 1$.

What is the difference from the solution of h_n from the previous question to the solution of H_n?

① α_1 is different

② there is an extra "-1" in the exact formula H_n

13. Definition: Linear Non-homogeneous Recurrence Relation with constant coefficients.

A linear non-homogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called associated homogeneous recurrence relation.

14. Find the associated homogeneous recurrence relation from the non-homogeneous recurrence relation.

a) $a_n = 2a_{n-1} + 2^n$: Its associated homogeneous recurrence relation is $a_n = 2a_{n-1}$.

b) $a_n = a_{n-1} + a_{n-2} + n^2 + 2n$: Its associated homogeneous recurrence relation is $a_n = a_{n-1} + a_{n-2}$.

c) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$: its associated homogeneous recurrence relation is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

15. Thm. 5: The Solution of a Linear Non-homogeneous Recurrence Relation with constant coefficients.

Given a Linear Non-homogeneous Recurrence Relation with constant coefficients:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

The solution of this recurrence relation includes two parts $a_n^{(h)}$ and $a_n^{(p)}$ where

$a_n^{(h)}$ is the solution of associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$,

$a_n^{(p)}$ is called a particular solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, and then the general solution is of the form $a_n^{(h)} + a_n^{(p)}$.

16. From 11. and 12., for the linear non-homogeneous recurrence relation $H_n = 2H_{n-1} + 1$ with $\boxed{H_1 = 1}$, find $H_n^{(h)}$ (the solution of its associated homogeneous recurrence relation) and $H_n^{(p)}$ (particular solution of this non-homogeneous recurrence relation). (Let $H_n^{(h)} = r^n, \ldots$)

For $H_n^{(h)}$, it satisfies $H_n = 2H_{n-1}$, $H_n^{(h)} = \alpha_1 (2)^n$

For $H_n^{(p)}$, we guess $H_n^{(p)} = A$ where A is an undetermined constant.

$H_{n-1}^{(p)} = A$ plugging into $H_n^{(p)} = 2H_{n-1}^{(p)} + 1$ and we get $A = 2A + 1$, $A = -1$

$H_n^{(p)} = -1$, Then we have $H_n = H_n^{(h)} + H_n^{(p)} = \alpha_1 (2)^n - 1$, since $H_1 = 1$

$1 = H_1 = \alpha_1 (2)^1 - 1 \Rightarrow 1 = 2\alpha_1 - 1 \Rightarrow 2 = 2\alpha_1 \Rightarrow \alpha_1 = 1$

$H_n = (2)^n - 1$