MAT2540, Classwork6, Spring2026

6.1 The Basics of Counting (Skip trees)

1. The **Product Rule** With m Tasks

Suppose that a procedure can be broken down into a sequence of m tasks $T_1, T_2, \cdots, T_m$. If each task T_i can be done in n_i ways, for $i = 1,2,\cdots m$, then there are $n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_m$ ways to do the procedure.

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26 10

2. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$

$f(x_1) = y_{100}$, $f(x_3) = y_{100}$

$x_3, x_5, x_1, x_2, \ldots, x_{100}$; $y_1, y_2, y_3, y_4, \cdots, y_n$; $x_4, x_3 \to y_{100}$

3. Example: Counting One-to-One Functions. $f(x) = y$, if f is 1-1, same output gets same input

How many one-to-one functions are there from a set with m elements (domain) to one with n elements? → codomain

If $m > n$, there are no one-to-one function from a set of m to a set of n elements

If $m \le n$,

m objects → y_1, y_2, y_3, ..., y_n : 1st object n ways $\cdot$ 2nd object $n-1$ ways $\cdot \cdots$ mth object $n-m+1$ ways

We have $n \cdot (n-1) \cdot (n-2) \cdots (n-m+1)$ ways of one-to-one function

4. Given a set S. Then the number of the elements in S is denoted by $|S|$.

$a \in \mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}$, $a(x, y)$, $\mathbb{R} \leftarrow x$, $y \to \mathbb{R}$

5. The product rule in terms of sets: If $A_1, A_2, \cdots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets $A_1 \times A_2 \times \cdots \times A_m$ is the product of the number of elements in each set:

$|A_1 \times A_2 \times A_3 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot |A_3| \cdot \cdots \cdot |A_m|$

6. Example: Counting Subsets of a Finite Set. $S = \{1, 2, 3\}$, $|S| = 3$, $2^3 = 8$: $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{\emptyset\}$

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Let $|S| = n$ be a finite number

$S = \{a_1, a_2, a_3, a_4, \cdots, a_n\}$, for each element of S, either you pick it or you don't

the ways to pick: $2 \cdot 2 \cdot 2 \cdot 2 \cdots \cdot 2 = 2^n = 2^{|S|}$

7. Please read the description about DNA and Genomes in Example 11 on p.409 before doing HW5.

8. The **Sum Rule** With m Tasks

Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,... , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \le i < j \le m$. Then the number of ways to do the task is $n_1 + n_2 + n_3 + \cdots + n_m$.

9. The sum rule in terms of sets: If $A_1, A_2, \cdots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m$ is the sum of the numbers of elements in the sets:

$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m| = |A_1| + |A_2| + |A_3| + \cdots + |A_m|$

10. Let $A_1 = \{1,2,3\}, A_2 = \{4,5,6,9\}$, and $A_3 = \{7,8\}$. Find $|A_1| =$ 3, $|A_2| =$ 4, $|A_3| =$ 2,

$A_1 \cup A_2 \cup A_3 =$ $\{1,2,3,4,5,6,9,7,8\}$ and $|A_1 \cup A_2 \cup A_3| =$ 9.

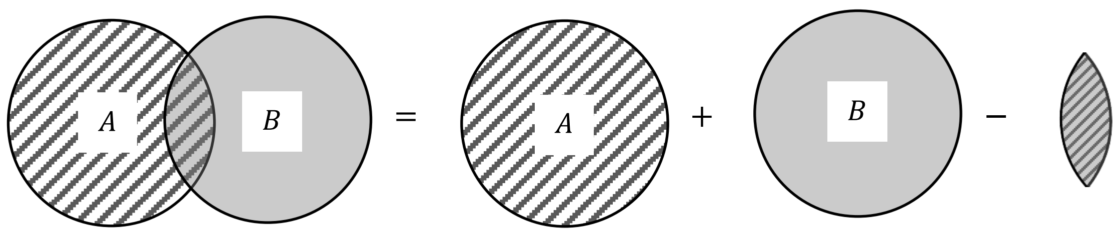
$|A_1| + |A_2| + |A_3| = 3+4+2 = 9$

11. The **Subtraction Rule**

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2 -$ the number of ways to do this task that are common to the two different ways.

12. The **Principle of Inclusion-Exclusion** for two sets:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be sets. We have $|A \cup B| = |A| + |B| - |A \cap B|$



13. Let $A = \{1,2,3\},\ B = \{1,3,5\}$. Show that $|A \cup B| = |A| + |B| - |A \cap B|$.

$A \cup B = \{1,2,3,5\} \Rightarrow |A \cup B| = 4$, $|A| = 3$. $|B| = 3$

$A \cap B = \{1,3\} \Rightarrow |A \cap B| = 2$ and $|A| + |B| - |A \cap B|$

$= 3+3-2 = 4 = |A \cup B|$

14. The Principle of Inclusion-Exclusion for **three** sets:

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be sets. Then we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

15. The **Division Rule**

There are $\frac{n}{d}$ ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

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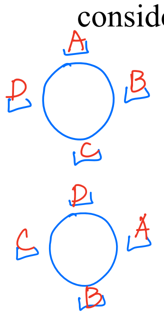
16. How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?



ABCD BACD BCAD ABDC ACBD BADC

BCDA ACDB CADB CABD CBDA CBAD

CDAB CDBA ADBC DCAB BDAC DCBA

DABC DBAC DBCA BDCA DACB ADCB

$n = 4 \cdot 3 \cdot 2 \cdot 1$, $d = 4$. total ways to seat four ppl around circular table is $\frac{n}{d} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4} = 6$