MAT2540, Classwork2, Spring2026

5.3 Recursive Definitions and Structural Induction Part 1 (p. 365-370)

1. Review of 2.4: Define a Sequence by **Recursive Relations**:

   Another popular method to define a sequence is to provide one or more initial terms together with a recursive rule for determining subsequent terms from those that precede them.

2. Let $\{a_n\}$ be a sequence that satisfies the **initial term** $a_0 = 2$ and the **recurrence relation**

$$a_n = a_{n-1} + 3 \text{ for } n = 1, 2, 3, \cdots. \quad a_0 = 2$$

   What are a_1, a_2, and a_3?

$$a_1 = a_0 + 3 = 2+3 = 5 \quad +3$$
$$a_2 = a_1 + 3 = 5+3 = 8 \quad +3$$
$$a_3 = a_2 + 3 = 8+3 = 11 \quad +3$$

   $\Rightarrow$ Arithmetic Sequence with a common difference $d = 3$

   Explicit formula of a_n:

$$a_n = a_0 + n \cdot d = 2 + 3n$$

3. Let $\{a_n\}$ be a sequence that satisfies the **initial term** $a_0 = 3$ and the **recurrence relation**

$$a_n = \frac{1}{3} a_{n-1} \text{ for } n = 1, 2, 3, \cdots.$$

   What are a_1, a_2, and a_3?

$$a_1 = \frac{1}{3} a_0 = \frac{1}{3} \cdot 3 = 1 \quad \times \frac{1}{3}$$
$$a_2 = \frac{1}{3} a_1 = \frac{1}{3} \cdot 1 = \frac{1}{3} \quad \times \frac{1}{3}$$
$$a_3 = \frac{1}{3} a_2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

   $\Rightarrow$ Geometric Sequence with a common ratio $r = \frac{1}{3}$

   Explicit formula of a_n:

$$\Rightarrow a_n = a_0 \cdot r^n = 3 \cdot \left(\frac{1}{3}\right)^n \quad \text{or} \quad \left(\frac{1}{3}\right)^{n-1}$$

4. (*Fibonacci sequence*) Let $\{f_n\}$ be a sequence that satisfies the **initial term** $f_0 = 0$, $f_1 = 1$, and **recurrence relation**

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \cdots.$$

   What are the first five terms?

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

   Explicit formula (also called a closed formula) of *Fibonacci sequence*:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$n = 1, 2, 3, 4, \cdots$$

5. Recursively Defined Function.

   We use two steps to define a function with the set of nonnegative integers as its domain:

   BASIS STEP: Specify the value of the function at zero.

   RECURSIVE STEP: Give a rule for finding its value at an integer from its value at smallest integers

   Such a definition is called a recursive or inductive definition.

6. The Recursive Sequences and the Induction.

   When we define a sequence recursively by specifying how terms of the sequence are found from previous terms, we can use induction to prove results about the sequence.

$\sqrt{4} = 2 < \sqrt{5} < 3 = \sqrt{9}$

7. Let $\{f_n\}$ be the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \cdots$.

Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, *where* $\alpha = (1+\sqrt{5})/2$. $\boxed{\text{(Hint: } \alpha^2 - \alpha - 1 = 0)}$ $\alpha^2 = \alpha + 1$

Recognize $P(n)$: $f_n > \alpha^{n-2}$ where $n \geq 3$, $\alpha = \frac{1+\sqrt{5}}{2}$

Basis Step. Show $P(3)$ and $P(4)$ are true:

$n=3$, $f_3 = 2 \overset{?}{>} \left(\frac{1+\sqrt{5}}{2}\right)^1$ Yes, $n=4$, $f_4 = 3 \overset{?}{>} \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$ (Yes)

Inductive Step:

Assume $P(j)$ is true ($f_j > \alpha^{j-2}$) with $3 \leq j \leq k$, where $k \geq 4$

To show $P(k+1)$ is true $f_{k+1} = f_k + f_{k-1}$

(To prove $f_{k+1} > \alpha^{k+1-2} = \alpha^{k-1}$)

$\alpha^{k-1} = \alpha^2 \cdot \alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$

$$\left(\begin{aligned} &\boxed{\alpha^2 - \alpha - 1 = 0} \\ &\alpha^{k-3}(\alpha^2 - \alpha - 1) = 0 \\ &\alpha^{k-1} - \alpha^{k-2} - \alpha^{k-3} = 0 \\ &\alpha^{k-1} = \alpha^{k-2} + \alpha^{k-3} \end{aligned} \right)$$

By inductive hypothesis, if $\boxed{k \geq 4}$, we have $f_k > \alpha^{k-2}$ and $f_{k-1} > \alpha^{k-3}$

Therefore, $\boxed{f_{k+1}} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = \boxed{\alpha^{k-1}}$

Hence, $P(k+1)$ is true and this completes the proof.

8. **The Euclidean Algorithm:** Let $a = qb + r$, where a, b, q, r are integers. Then gcd(a, b) = gcd(b, r).

9. Find GCD by using the Euclidean Algorithm. Let $d = \gcd(24, 36)$. We have $d|24$ and $d|36$.

36 $=$ $\underline{1}$ $\times$ **24** $+$ $\underline{12}$, then $\underline{36}$ $=$ $\underline{12}$ **mod** $\underline{24}$ and $d|\underline{12}$. It implies $d = \gcd(24, \underline{12})$.

24 $=$ $\underline{2}$ $\times$ **12** $+$ $\underline{0}$, then $\underline{24}$ $=$ $\underline{0}$ **mod** $\underline{12}$ and $d|\underline{0}$. It implies $d = \gcd(12, \underline{0})$. Thus, $d = \underline{12}$.

10. (*LAMÉ's Theorem*) Let a and b be positive integers with $a \geq b$. Then the number of divisions used by the Euclidean algorithm to find gcd(a, b) is less than or equal to five times the number of decimal digits in b.

$r_0 = r_1 q_1 + r_2$, $0 \leq r_2 < r_1$ $(a = r_0, b = r_1)$

$r_1 = r_2 q_2 + r_3$, $0 \leq r_3 < r_2$

$\vdots$

$r_{n-2} = r_{n-1} q_{n-1} + r_n$, $0 \leq r_n < r_{n-1}$

$r_{n-1} = r_n q_n + 0$ (where the remainder is 0)

Then $\gcd(a,b) = r_n$. Here we know

(1) n divisions have been used to find $\gcd(a,b)$

(2) $q_1, q_2, q_3, \cdots, q_n$ are at least 1

(3) since $r_n < r_{n-1}$, $q_n \geq 2$

Using the Fibonacci sequence $\{f_n\}$, we have

$r_n \geq 1 \overset{= f_2}{} \Rightarrow r_n \geq f_2$,

$r_{n-1} = q_n r_n \geq 2 \cdot r_n \geq 2 = f_3$

$r_{n-2} = q_{n-1} r_{n-1} + r_n \geq r_{n-1} + r_n \geq f_3 + f_2 = f_4$

$\vdots$

$r_2 = r_3 q_3 + r_4 \geq r_3 + r_4 \geq f_{n-1} + f_{n-2} = f_n$

$b = r_1 = r_2 q_2 + r_3 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}$

It implies $b \geq f_{n+1} > \alpha^{n-1}$ by question 7.

Here $b > \alpha^{n-1} > 0$ take $\log_{10}$ $\log_{10}(b) > (n-1)\log_{10}(\alpha)$

$n - 1 < \frac{\log_{10}(b)}{\log_{10}(\alpha)} \Rightarrow n < \frac{\log_{10}(b)}{\log_{10}(\alpha)} + 1 \approx 5 \cdot \log_{10}(b) + 1$