MAT2540, Classwork1, Spring2026

5.2 Strong Induction and Well-Ordering

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

1. **Mathematical Induction**. To prove that $P(n)$ is true for all positive integers n, we complete 2 steps.

   BASIS STEP: We verify that $P(1)$ is true.

   INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

2. An Example of Proof using Mathematical Induction: *Prove that $n^2 + n$ is divisible by 2 for all $n \in \mathbb{Z}^+$.* (belongs to)

   Recognize $P(n)$: n^2+n is divisible by 2 (for all $n \in \mathbb{Z}^+$)

   Basis Step. Show $P(1)$ ($n=1$) is true: Is $1^2+1=2$ divisible by 2? Yes, it is true

   Inductive Step. Assume $P(k)$ is true for $k>0$, then prove $P(k+1)$ is true

   k^2+k is divisible by 2 $\Rightarrow \frac{k^2+k}{2} = m$ where m is an integer $\Rightarrow k^2+k=2m$

   Then we have $(k+1)^2+(k+1) = k^2+2k+1+k+1 = k^2+3k+2$

   $= \boxed{k^2+k}+2k+2 = 2m+2k+2 = 2(m+k+1) \Rightarrow (k+1)^2+(k+1)$ is divisible by 2

   Conclusion: $P(n)$ is true for all $n \in \mathbb{Z}^+$

3. **Strong Induction**. To prove that $P(n)$ is true for all positive integers n, we complete 2 steps:

   BASIS STEP: We verify that $P(1)$ is true.

   INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

4. An Example of Proof using Strong Induction.

   Show that if n is an integer greater than 1, then n can be written as the product of primes.

Recognize $P(n)$: if n is an integer greater than 1, then n can be written as the product of primes

Basis Step. Show $P(2)$ is true: Because $2 = 1 \times 2$, this is true.

Inductive Step: Assume $P(j)$ is true for $2 \le j \le k$, it means any j can be a product of primes. To show that $P(k+1)$ is true, that is, $k+1$ is the product of primes. There are two cases:

(1) If $k+1$ is prime. we immediately see that $P(k+1)$ is true.

(2) If $k+1$ is composite, it means $(k+1) = a \cdot b$, $2 \le a \le b \le k$. We can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1 = a \cdot b$, then $k+1$ is a product of primes. It means that $P(k+1)$ is true.

5. How to decide which induction to use?

6. **Modified Strong Induction**

Let b be a fixed integer. To prove that $P(n)$ is true for all positive integers n with $n \geq b$, we complete 2 steps:

BASIS STEP: We verify that the propositions $P(b), P(b+1), \ldots, b(j)$ are true where j is a fixed positive integer and $j > b$.

INDUCTIVE STEP: We show that the conditional statement $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for every positive integer $k > j$.

7. Prove that *every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps*.

(a) Prove the statement by **Mathematical Induction**.

Basis Step. Show $P(12)$ is true: Since $12 = 4+4+4$, then $P(12)$ is true.

Inductive Step. Assume $P(k)$ is true for $k \geqslant 12$. Then to prove $P(k+1)$ is true, we have 2 cases:

(1) k-cent is formed using at least one 4-cent stamp: $k = c+4$ for an integer c which is formed by 4-cent and 5-cent stamps, then $k+1 = c+5$ which means $P(k+1)$ is true.

(2) k-cent is formed without 4-cent stamps: $k \geqslant 15$. $k = c+5+5+5$ where c is formed by 5-cent stamp(s), then $k+1 = c+4+4+4+4$ which means $(k+1)$-cent is also formed by 4-cent and 5-cent stamps.

Conclusion: By (1), (2), $P(k) \to P(k+1)$ and $P(n)$ is true for $n \geqslant 12$

(b) Prove the statement by **Strong Induction**. Let $b = 12$, $j = 15$ (why?)

Basis Step. Show $P(12), P(13), P(14), P(15)$ ~~P(16)~~ are true: Since we have $12 = 4+4+4$, $13 = 4+4+5$, $14 = 4+5+5$, $15 = 5+5+5$, then this shows all four of them are true.

Inductive Step. Assume $P(j)$ is true, $12 \leq j \leq k$, where k is an integer with $k \geqslant 15$.

~~$k \geqslant 12$~~

To prove $P(k+1)$ is true, we do the following:

By assumption that $P(k-3)$ is true because $k-3 \geqslant 12$: we can form $(k-3)$-cent into 4- and 5-cent stamps.

For $P(k+1)$, we have $k+1 = k-3+4$, which means $k+1$ can be formed by $(k-3)$-cent and a 4-cent stamp. Therefore, $P(k+1)$ is true and this completes the inductive step.

8. The Well-Ordering Property: Every nonempty set of non-negative integers has a least element.

Example: $A = \{5, 8, 3, 11\}$. The least element is 3.

9. The Equivalent Principles.

The validity of both math induction and strong induction follow from the well-ordering property (Appendix 1.)