MAT1375, Classwork17, Fall2026

Ch16. Compound interest and half-life

1. The function to describe the population growths or decline:

$$f(t) = c \cdot b^t,$$

where $f(0) =$ ___c___ which is initial condition at $t =$ ___0___, and $b > 0, b \neq 1$.

If the **rate of growth** r is given, then the base $b =$ ___e^r___ and $f(t) = c \cdot (___)^t$. ① ② $= c \cdot e^{rt}$

2. The population size of a country was 12.7 million in the year 2010, and 14.3 million in the year 2020.

(a) Assuming an exponential growth for the population size, find the formula for the population depending on the year t.

Exponential growth $P(t) = C \cdot b^t$

year 2010 $t=0$, 12.7 (m), $P(0) = C \cdot b^0 = 12.7 \Rightarrow C = 12.7$

year 2020 $t=10$ 14.3 (m), $P(10) = C \cdot b^{10} = 14.3$

$\Rightarrow 12.7 \cdot b^{10} = 14.3$

$\times \frac{1}{12.7}$ on both sides

$b^{10} = \frac{14.3}{12.7}$

(Take $\sqrt[10]{\cdot}$) $\sqrt[10]{b^{10}} = \sqrt[10]{\frac{14.3}{12.7}}$

$\Rightarrow b = \sqrt[10]{\frac{14.3}{12.7}} = \left(\frac{14.3}{12.7}\right)^{\frac{1}{10}}$

$P(t) = 12.7 \cdot \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}$

(b) What will the population size be in the year 2026, assuming the formula holds until then?

Year 2026 $\Rightarrow t = 2026 - 2010 = 16$

the population in 2026 is $P(16) = 12.7 \cdot \left(\frac{14.3}{12.7}\right)^{\frac{16}{10}}$

(c) When will the population reach 18 million?

$P(?) = 18 \Rightarrow 18 = 12.7 \cdot \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}$

$\frac{18}{12.7} = \left(\frac{14.3}{12.7}\right)^{\frac{t}{10}}$

Take "ln" on the both sides

$\ln\left(\frac{18}{12.7}\right) = \frac{t}{10} \cdot \ln\left(\frac{14.3}{12.7}\right)$

$t = 10 \cdot \frac{\ln\left(\frac{18}{12.7}\right)}{\ln\left(\frac{14.3}{12.7}\right)} = 2.94$

Year $2010 + 10 \cdot \frac{\ln\left(\frac{18}{12.7}\right)}{\ln\left(\frac{14.3}{12.7}\right)} \approx 2039$

3. The population of a country grows exponentially at a rate of 2.6% per year. If the population was 35.7 million in the year 2000, then what is the population size of this country in the year 2027?

$b = e^r$, $r = 2.6\% = 0.026 \Rightarrow b = e^{0.026}$

Year 2000 $P(0) = 35.7$, $\rightarrow C = 35.7$

$t = 0$,

$P(t) = C \cdot b^t = 35.7 \cdot e^{0.026t}$

In Year 2027, the population $P(27) = 35.7 \cdot e^{0.026 \cdot 27}$

$t = 2027 - 2000 = 27$

$(e = 2.718)$

4. If a principal (i.e. initial amount) ① ***P*** is invested for ***t*** years at a ② rate ***r*** and ③ compounded ***n*** times per year, then the final amount ***A*** is given by

$$A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{nt}.$$

If a **continuous compounding** ($n \to$ ∞) is considered, then the final amount ***A*** with a continuous compounding is given by $A(t) = P \cdot \left(e^r\right)^t$.

$A(t) = 500\left(1 + \frac{0.03}{4}\right)^{4 \cdot t}$

5. How much do you get if you invest $500 today at 3% compounded quarterly in 3 years?

① $P = \$500$

② $r = 3\% = 0.03$

③ $n = 4$ (how many quarters in a year)

④ $t = 3$

$A(3) = 500 \cdot \left(1 + \frac{0.03}{4}\right)^{4 \cdot 3} = 500\left(1 + \frac{0.03}{4}\right)^{12}$

6. The exponential function $f(t)$ has a half-life of ***h*** if the base is given by

$$b = \left(\frac{1}{2}\right)^{\frac{1}{h}},$$

and $f(t) = c \cdot \left(\left(\frac{1}{2}\right)^{\frac{1}{h}}\right)^t = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$.

7. The half-life of carbon-14 is 5730 years. If a dead tree trunk has 86% of its original carbon-14. how many years ago did the tree die?

$h = 5730$,

$f(t) = c\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$f(0) = c\left(\frac{1}{2}\right)^{\frac{0}{5730}} = c$

$86\% \cdot c = f(t) = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\Rightarrow 0.86 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\ln(0.86) = \ln\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\ln(0.86) = \frac{t}{5730}\ln\left(\frac{1}{2}\right)$

$\Rightarrow \frac{\ln(0.86)}{\ln(0.5)} = \frac{t}{5730} \Rightarrow t = 5730 \cdot \frac{\ln(0.86)}{\ln(0.5)}$