

Ch9. Roots of Polynomials

1. Factors and roots of polynomials.

Every n -degree polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, ($a_n \neq 0$) can be factored as

$$f(x) = a_n (x-c_1)(x-c_2)(x-c_3)(x-c_4) \dots (x-c_n)$$

Thus, the polynomial $f(x)$ of degree n has exactly n roots (which are $c_1, c_2, \dots, c_n$) and these roots may be either real or complex.
 (e.g. $3+2i, 3-2i, \sqrt{2}+5i, \sqrt{2}-5i, 3+4i, 3-4i$)

2. The Repeat roots and its **Multiplicity**.

Let $f(x) = (x-r)^n$ where r is the root of f and this root repeats n times. We call r a root with multiplicity n .

3. The complex root and its Conjugate.

Let f be a polynomial with all **real coefficients**. The complex roots are always found as a **pair**, that is, if $c = a + bi$ is a complex root of f , then the complex conjugate $\bar{c} = a - bi$ is also a root of f .
 (e.g. $c = 3+5i, \bar{c} = 3-5i$)

4. The Relation between Roots and Coefficient a_0 .

For a n -degree polynomial $f(x) = 1x^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$, we can factorize it as

$$f(x) = 1(x-c_1)(x-c_2)(x-c_3)(x-c_4) \dots (x-c_n)$$

where $c_1, c_2, \dots, c_n$ are the roots of $f(x)$. Then we have

$$(1) f(c_1) = 0, f(c_2) = 0, f(c_3) = 0, \dots, f(c_n) = 0$$

$$(2) a_0 = (-c_1) \cdot (-c_2) \cdot (-c_3) \cdot (-c_4) \cdot \dots \cdot (-c_n)$$

(3) The factor of a_0 might be the possible candidates for the root of $f(x)$.

5. Let $f(x) = x^3 - x^2 + 2$. Find all the roots of $f(x)$.

Step 1 guess for roots of $f(x)$ which is the factors of 2 : $\pm 1, \pm 2$

$$\text{Step 2 } f(1) = (1)^3 - (1)^2 + 2 = 1 - 1 + 2 = 2 \neq 0$$

$$f(-1) = (-1)^3 - (-1)^2 + 2 = -1 - 1 + 2 = 0 \Rightarrow x = -1 \text{ is a root of } f(x)$$

$\Rightarrow (x+1)$ is a factor of $f(x)$

$$\text{Step 3 } f(x) = (x+1)(x^2 - 2x + 2)$$

Step 4 Find the root of $f(x)$...

$$(x+1)(x^2 - 2x + 2) = 0 \Rightarrow (x+1) = 0 \text{ or } (x^2 - 2x + 2) = 0$$

$$\Rightarrow x = -1 \text{ or } x = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow x = -1 \text{ or } x = \frac{2 \pm 2i}{2}$$

$$\Rightarrow x = -1 \text{ or } x = 1 \pm i$$

3 roots: $x = -1, x = 1+i, x = 1-i$

(Real) & (a pair of the complex roots)

$$\begin{array}{r} x^3 - x^2 + 0x + 2 \\ \underline{-(x^3 + x^2)} \\ -2x^2 + 0x + 2 \\ \underline{-(-2x^2 - 2x)} \\ 2x + 2 \\ \underline{-(2x + 2)} \\ 0 \end{array}$$

$\frac{x^3}{x} = x^2$
 $\frac{-2x^2}{x} = -2x$
 $\frac{2x}{x} = 2$

$$\begin{aligned} 2x^2 + 3x - 2 &= (x+2)(2x-1) \\ &= 2(x+2)(x-\frac{1}{2}) \\ &= 2(x-(-2))(x-\frac{1}{2}) \end{aligned}$$

$$f(x) = x^3 + 2x^2 + 5x + 4$$

factors of 4 are $\pm 1, \pm 2, \pm 4$

6. The Number Line Test: Given a polynomial $f(x)$

Step 1. Solve the equation $f(x)=0$ and find all the roots.

Step 2. Mark the roots on the number line and check sign (positivity/negativity) in each subinterval.

7. Let $f(x) = x^3 - 3x^2 + 4$. Find all the roots of $f(x)$. Sketch a complete graph and label all roots.

① Possible roots: the factors of 4: $\pm 1, \pm 2, \pm 4$

$$f(1) = (1)^3 - 3(1)^2 + 4 = 1 - 3 + 4 = 2 \neq 0$$

$$f(-1) = (-1)^3 - 3(-1)^2 + 4 = -1 - 3 + 4 = 0 \checkmark$$

$x = -1$ is a root of $f(x)$, $(x+1)$ is a factor of $f(x)$

② long division

$$\begin{array}{r} x^2 - 4x + 4 \\ x+1 \overline{) x^3 - 3x^2 + 0x + 4} \\ \underline{x^3 + x^2} \\ -4x^2 + 0x \\ \underline{-4x^2 + 4x} \\ 4x + 4 \\ \underline{4x + 4} \\ 0 \end{array}$$

③ Find the roots

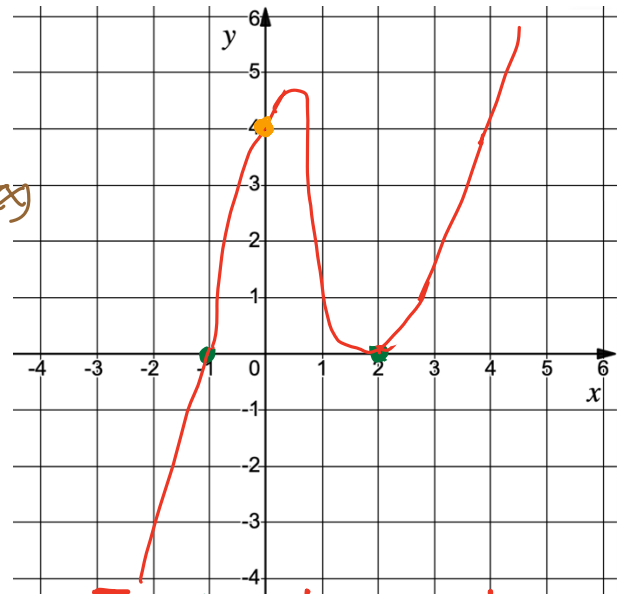
$$f(x) = (x+1)(x^2 - 4x + 4)$$

$$x^2 - 4x + 4 = (x-2)^2$$

$$= (x+1)(x-2)(x-2) = 0$$

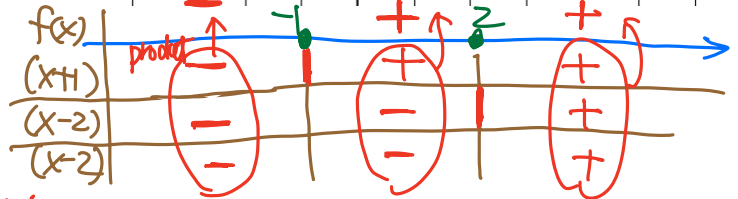
$$\Rightarrow (x+1)=0 \text{ or } (x-2)=0 \text{ or } (x-2)=0$$

roots: $x = -1, x = 2, x = 2$



④ Number line test

- Mark the roots on number line
- List all factors of $f(x)$
- try the points to check +/



8. Let $f(x) = x^3 - x^2 - 9x + C$ where C is a real number. If $x = 3$ is a root of $f(x)$, find C so that $f(x)$ has this root as indicated. Then, for this choice of C , find all remaining roots of $f(x)$.

$$f(3) = 0, C = ?$$

$$f(3) = 3^3 - 3^2 - 9 \cdot 3 + C = 0$$

$$\Rightarrow 27 - 9 - 27 + C = 0$$

$$\Rightarrow -9 + C = 0 \Rightarrow C = 9$$

(implies $(x-3)$ is a factor of $f(x)$)

$$f(x) = x^3 - x^2 - 9x + 9$$

$$= x^2(x-1) - 9(x-1)$$

$$= (x-1)(x^2 - 9)$$

$$= (x-1)(x+3)(x-3)$$

$$f(x) = (x-1)(x+3)(x-3) = 0$$

$$(x-1)=0 \text{ or } x+3=0 \text{ or } x-3=0$$

$$x=1 \text{ or } x=-3 \text{ or } x=3$$

$$x-3 \overline{) x^2 + 0x - 9}$$

9. Find a polynomial $f(x)$ that fits the given data.

$f(x)$ has degree 4. $f(x)$ has roots 0, 2, -1, -4, and $f(1) = 20$.

The factors of $f(x)$ are

$$x(x-2)(x+1)(x+4)$$

$$f(x) = -2 \cdot x(x-2)(x+1)(x+4)$$

$$f(1) = (-2) \cdot 1 \cdot (1-2) \cdot (1+1) \cdot (1+4)$$

$$= (-2) \cdot 1 \cdot (-1) \cdot (2) \cdot (5) = (-2) \cdot (-10) = 20$$

