Mat 1575 HW12 (§5.1)

Find the first six terms of each of the following sequences, starting with $n = 1$.

1. $a_n = 1 + (-1)^n$ for $n \geq 1$

$a_1 = 1+(-1)^1 = 1-1 = 0$, $a_2 = 1+(-1)^2 = 1+1 = 2$, $a_3 = 1+(-1)^3 = 1-1 = 0$,
$a_4 = 1+(-1)^4 = 1+1 = 2$, $a_5 = 1+(-1)^5 = 1-1 = 0$, $a_6 = 1+(-1)^6 = 1+1 = 2$

first six terms: $0, 2, 0, 2, 0, 2$

3. $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

$a_1 = 1$, $a_2 = a_1 + 2 = 1+2 = 3$, $a_3 = a_2 + 3 = 3+3 = 6$,
$a_4 = a_3 + 4 = 6+4 = 10$, $a_5 = a_4 + 5 = 10+5 = 15$, $a_6 = a_5 + 6 = 15+6 = 21$

first six terms: $1, 3, 6, 10, 15, 21$

7. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.

First term $a_1 = -3$,
Common difference $d = a_{n+1} - a_n = 4$
$\Rightarrow$ Explicit term $a_n = a_1 + (n-1)\cdot d$
$= -3 + (n-1)\cdot 4$
$= 4n - 7$

$a_n = 4n - 7$

9. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

First term $a_1 = 3$.
Common ratio $r = \frac{a_{n+1}}{a_n} = 10$
$\Rightarrow$ Explicit term $a_n = a_1 r^{n-1}$
$= 3 \cdot 10^{n-1}$

$a_n = 3 \cdot 10^{n-1}$

Find a formula for the general term a_n of each of the following sequences.

12. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ (*Hint:* Find where $\sin x$ takes these values)

$a_1 = \sin(\frac{\pi}{2}\cdot 1) = 1$, $a_2 = \sin(\frac{\pi}{2}\cdot 2) = \sin(\pi) = 0$

$a_3 = \sin(\frac{\pi}{2}\cdot 3) = -1$, $a_4 = \sin(\frac{\pi}{2}\cdot 4) = 0$

$a_5 = \sin(\frac{\pi}{2}\cdot 5) = 1$ . . . . . $\Rightarrow a_n = \sin(\frac{\pi}{2}\cdot n)$

13. $\{1, -1/3, 1/5, -1/7, \ldots\}$ odd number $1, 3, 5, 7, \cdots, (2n-1)$

Alternative property: $+, -, +, -, +, \ldots$ $(-1)^{n+1}$

$\Rightarrow a_n = \frac{(-1)^{n+1}}{2n-1}$ or $\frac{(-1)^{n-1}}{2n-1}$

Find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

15. $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

First term $a_1 = 2$.

Common ratio $r = \frac{a_{n+1}}{a_n} = 2$

$a_n = a_1 r^{n-1} = 2\cdot(2)^{n-1} = 2^n$

Suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n. Evaluate each of the following limits, or state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23. $\lim_{n\to\infty} (3a_n - 4b_n) = 3\lim_{n\to\infty} a_n - 4\lim_{n\to\infty} b_n = 3(1) - 4(-1) = 3 + 4 = 7$

25. $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} = \frac{\lim_{n\to\infty}(a_n + b_n)}{\lim_{n\to\infty}(a_n - b_n)} = \frac{\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n}{\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n} = \frac{1-1}{1-(-1)} = \frac{0}{2} = 0$

Find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27. $\frac{n^2}{2^n} \Rightarrow \lim_{n\to\infty} \frac{n^2}{2^n} \overset{L'}{=} \underset{\text{type}}{\left(\frac{\infty}{\infty}\right)} \lim_{n\to\infty} \frac{2n}{\ln 2 \cdot 2^n} \overset{L'}{=} \underset{\text{type}}{\left(\frac{\infty}{\infty}\right)} \lim_{n\to\infty} \frac{2}{(\ln 2)^2 2^n} = 0$ ($2^n \to \infty$)

29. $\frac{\sqrt{n}}{\sqrt{n+1}} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = \frac{1}{1} = 1$

For each of the following sequences, whose nth terms are indicated, state whether the sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31. $n/2^n, n \geq 2$ $\Rightarrow$ $n<2^n \Rightarrow 0<\frac{n}{2^n}<\frac{2^n}{2^n} \Rightarrow 0<\frac{n}{2^n}<1 \Rightarrow$ bounded

$a_1=\frac{1}{2}, a_2=\frac{2}{4}, a_3=\frac{3}{8}, a_4=\frac{4}{16}=\frac{1}{4}, a_5=\frac{5}{32}, a_6=\frac{6}{64}$

$a_1=a_2>a_3>a_4>a_5>a_6>\cdots$, it's decreasing

33. $\sin n$ $\Rightarrow$ $-1 \leq \sin(n) \leq 1 \Rightarrow$ It's bounded.

but $\sin(n)$ is NOT MONOTONE.

35. $n^{1/n}, n \geq 3$ $\Rightarrow$ $1=1^{\frac{1}{n}}<n^{\frac{1}{n}}=e^{\frac{1}{n}\ln(n)} \leq e^1 \Rightarrow$ bounded.

$a_1=1^1=1, a_2=2^{\frac{1}{2}}=\sqrt{2}=1.4\cdots, a_3=3^{\frac{1}{3}}=1.\cdots \Rightarrow$ increasing

37. $\tan n$ $\Rightarrow$ $\tan(n) \in (0,\infty) \Rightarrow$ It's not bounded but increasing

Determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

47. $a_n = (2n)^{1/n} - n^{1/n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2^{\frac{1}{n}} \cdot n^{\frac{1}{n}} - n^{\frac{1}{n}}\right) = 1\cdot 1 - 1 = 0$

$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n)} = \lim_{n\to\infty} e^{\frac{\ln(n)}{n}} = e^{\lim_{n\to\infty}\frac{\ln(n)}{n}} \overset{L'}{\underset{(\frac{\infty}{\infty})}{=}} e^{\lim_{n\to\infty}\frac{1}{n}} = e^0 = 1$

$\lim_{n\to\infty} 2^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(2)} = e^0 = 1$

49. $a_n = \left(1-\frac{2}{n}\right)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-\frac{2}{n})^n = \lim_{n\to\infty} e^{\ln(1-\frac{2}{n})^n} = \lim_{n\to\infty} e^{n\ln(1-\frac{2}{n})}$

$= e^{\lim_{n\to\infty}\frac{\ln(\frac{n-2}{n})}{\frac{1}{n}}} \overset{L'}{\underset{(\frac{0}{0})}{=}} e^{\lim_{n\to\infty}\frac{\frac{1}{n-2}-\frac{1}{n}}{-\frac{1}{n^2}}} = e^{\lim_{n\to\infty}\frac{\frac{2}{n(n-2)}}{-\frac{1}{n^2}}} = e^{\lim_{n\to\infty}\frac{-2n^2}{n(n-2)}} = e^{-2}$

51. $a_n = \dfrac{2^n+3^n}{4^n}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2^n+3^n}{4^n} = \lim_{n\to\infty} \left(\dfrac{2^n}{4^n} + \dfrac{3^n}{4^n}\right)$

$= \lim_{n\to\infty} \left(\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n\right) = 0+0 = 0.$