Mat 1575 HW 11 (§3.7)

Evaluate the following integrals. If the integral is not convergent, answer "divergent."

347. $\displaystyle\int_2^4 \frac{dx}{(x-3)^2}$ $= \int_2^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2} = \lim_{t\to 3^-}\int_2^t \frac{dx}{(x-3)^2} + \lim_{t\to 3^+}\int_t^4 \frac{dx}{(x-3)^2}$

when $x=3$, $\frac{1}{(x-3)^2}\to\infty$

so this is an improper integral

$\int_2^t \frac{dx}{(x-3)^2} = \left[\frac{-1}{(x-3)}\right]_2^t = \left[\frac{-1}{t-3} - \left(\frac{-1}{2-3}\right)\right] = \left[\frac{-1}{t-3} - 1\right]$

Then $\lim_{t\to 3^-}\int_2^t \frac{dx}{(x-3)^2} = \lim_{t\to 3^-}\left[\frac{-1}{t-3} - 1\right] \to \infty$, so $\int_2^3 \frac{dx}{(x-3)^2}$ diverges and

So do $\int_2^4 \frac{dx}{(x-3)^2}$

349. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-}\int_0^t \frac{dx}{\sqrt{4-x^2}} = \lim_{t\to 2^-}\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^t = \lim_{t\to 2^-}\left[\sin^{-1}\left(\frac{t}{2}\right) - \sin^{-1}(0)\right]$

$= \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$.

When $x\to 2$, $\sqrt{4-x^2}\to\infty$

So it's an improper one

Find $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int d\theta = \theta + C$

$x = 2\sin\theta$; $dx = 2\cos\theta\,d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$\Rightarrow \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{2}\right)$

351. $\displaystyle\int_1^{\infty} xe^{-x}\,dx$ ↑ improper

$= \lim_{t\to\infty}\int_1^t xe^{-x}dx = \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^t = \lim_{t\to\infty}\left[-te^{-t} - e^{-t} - (-e^{-1} - e^{-1})\right]$

$= 0 + 0 + 2e^{-1} = \boxed{\frac{2}{e}}$.

ILATE

Find $\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$

$u = x$ ⟍ $dv = e^{-x}dx$

$du = dx$ ⟋ $v = -e^{-x}$

$\lim_{t\to\infty}\left[-te^{-t}\right] \overset{L'}{=} \lim_{t\to\infty}\frac{-t}{e^t} = \lim_{t\to\infty}\frac{-1}{e^t} = 0$

($\infty\cdot 0$ type)

$\lim_{t\to\infty}\left[-e^{-t}\right] = \lim_{t\to\infty}\left[\frac{-1}{e^t}\right] = 0$

353. Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Since $\int_1^\infty \frac{dx}{\sqrt{x^3+1}} < \int_1^\infty \frac{dx}{\sqrt{x^3}}$, then if $\int_1^\infty \frac{dx}{\sqrt{x^3}}$ converges, so does $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$.

Check $\int_1^\infty \frac{dx}{\sqrt{x^3}} = \lim_{t\to\infty}\int_1^t x^{-\frac{3}{2}}dx = \lim_{t\to\infty}\left[-2x^{-\frac{1}{2}}\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t}} - \left(\frac{-2}{\sqrt{1}}\right)\right] = \lim_{t\to\infty}\frac{-2}{\sqrt{t}} + 2 = 2$ ($\frac{-2}{\sqrt{t}} \to 0$)

Thus, $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

355. $\int_0^\infty e^{-x}\cos x\,dx$

ILATE

Find $\int e^{-x}\cos(x)\,dx = -e^{-x}\cos(x) - \int(-e^{-x})(-\sin(x))\,dx = -e^{-x}\cos(x) - \left[-e^{-x}\sin(x) - \int -e^{-x}\cos(x)\,dx\right]$

$= -e^{-x}\cos(x) + e^{-x}\sin(x) - \int e^{-x}\cos(x)\,dx$

E: $u = \cos(x)$, $dv = e^{-x}dx$; $du = -\sin(x)dx$, $v = -e^{-x}$

T: $u = \sin(x)$, $dv = e^{-x}dx$; $du = \cos(x)dx$, $v = -e^{-x}$

$\Rightarrow 2\int e^{-x}\cos(x)\,dx = e^{-x}\sin(x) - e^{-x}\cos(x) + C \Rightarrow \int e^{-x}\cos(x)\,dx = \frac{e^{-x}\sin(x) - e^{-x}\cos(x)}{2} + C$

$\int_0^\infty e^{-x}\cos(x)\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\cos(x)\,dx = \lim_{t\to\infty}\left[\frac{e^{-x}\sin(x) - e^{-x}\cos(x)}{2}\right]_0^t$

$= \lim_{t\to\infty}\left[\frac{e^{-t}\sin(t) - e^{-t}\cos(t)}{2} - \left(\frac{e^0\sin(0) - e^0\cos(0)}{2}\right)\right]$

$= \frac{0}{2} - \frac{0}{2} - \frac{1\cdot 0}{2} + \frac{1\cdot 1}{2} = \boxed{\frac{1}{2}}$

$\lim_{t\to\infty}\frac{-1}{e^t} \le \lim_{t\to\infty} e^{-t}\sin(t) \le \lim_{t\to\infty}\frac{1}{e^t}$, ($\to 0$, $\to 0$)

since $\lim_{t\to\infty}\frac{-1}{e^t} = 0 = \lim_{t\to\infty}\frac{1}{e^t}$, then $\lim_{t\to\infty} e^{-t}\sin(t) = 0$

Similarly, $\lim_{t\to\infty} e^{-t}\cos(x) = 0$

357. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\ln x - 4\sqrt{x}\right]_t^1$

ILATE

as $x\to 0$, $\frac{\ln x}{\sqrt{x}} \to \frac{-\infty}{0}$

Find $\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\cdot\ln x - \int\frac{2\,dx}{\sqrt{x}} = 2\sqrt{x}\cdot\ln x - 4\sqrt{x} + C$

$u = \ln x \quad dv = \frac{dx}{\sqrt{x}}$

$du = \frac{dx}{x} \quad v = 2\sqrt{x}$

$= \lim_{t\to 0^+}\left[2\sqrt{1}\cdot\underbrace{\ln 1}_{=0} - 4\sqrt{1} - (2\sqrt{t}\cdot\ln(t) - 4\sqrt{t})\right]$

$\lim_{t\to 0^+} 2\sqrt{t}\cdot\ln(t) \overset{L'}{=}_{(0\cdot\infty \text{ type})} \lim_{t\to 0^+}\frac{2\ln(t)}{\frac{1}{\sqrt{t}}}$

$= [0 - 4 - 0 + 0] = \boxed{-4}$

$\overset{L'}{=}_{(\frac{\infty}{\infty})} \lim_{t\to 0^+}\frac{\frac{2}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}} = \lim_{t\to 0^+}\frac{2}{t}\cdot\frac{2t^{\frac{3}{2}}}{1} = \lim_{t\to 0^+} 4t^{\frac{1}{2}} = 0$

359. $\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx = \lim_{t\to-\infty}\int_t^0\frac{dx}{x^2+1} + \lim_{t\to\infty}\int_0^t\frac{dx}{x^2+1}$

$= \lim_{t\to-\infty}\left[\tan^{-1}(x)\right]_t^0 + \lim_{t\to\infty}\left[\tan^{-1}(x)\right]_0^t = \lim_{t\to-\infty}\left[\tan^{-1}(0) - \tan^{-1}(t)\right] + \lim_{t\to\infty}\left[\tan^{-1}(t) - \tan^{-1}(0)\right]$

$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \boxed{\pi}$

$\tan^{-1}(0) = 0$, $\lim_{t\to\infty}\tan^{-1}(t) = \frac{\pi}{2}$, $\lim_{t\to-\infty}\tan^{-1}(t) = \frac{-\pi}{2}$

361. $\int_{-2}^{2}\frac{dx}{(1+x)^2} = \lim_{t\to -1^-}\int_{-2}^{t}\frac{dx}{(1+x)^2} + \lim_{t\to -1^+}\int_t^2\frac{dx}{(1+x)^2}$

As $x\to -1$, $\frac{1}{(1+x)^2}\to\infty$

$= \lim_{t\to -1^-}\left[\frac{-1}{(1+x)}\right]_{-2}^{t} + \lim_{t\to -1^+}\left[\frac{-1}{(1+x)}\right]$

Since $\lim_{t\to -1^-}\left[\underbrace{\frac{-1}{1+t}}_{\to\infty} - \left(\frac{-1}{1-2}\right)\right]$ diverges, then $\lim_{t\to -1^-}\int_{-2}^{t}\frac{dx}{(1+x)^2}$ diverges, so does $\int_{-2}^{2}\frac{2\,dx}{(1+x)^2}$

363. $\int_0^{\infty} \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left[-\cos(x)\right]_0^t = \lim_{t\to\infty}\left[-\cos(t)-(-\cos 0)\right]$

$= \lim_{t\to\infty}\left[-\cos(t)+1\right]$

Although $\cos(x)$ is bounded, $\cos(x)$ is not monotone, then $\lim_{t\to\infty}\cos(t)$ diverges.

It implies $\lim_{t\to\infty}\left[-\cos(t)+1\right]$ diverges, so does $\int_0^{\infty}\sin x\,dx$.

365. $\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_t^1 = \lim_{t\to 0^+}\left[\frac{3}{2}\cdot 1 - \frac{3}{2}t^{\frac{2}{3}}\right] = \frac{3}{2}$ ($t^{\frac{2}{3}} \to 0$)

As $x\to 0$, $\frac{1}{\sqrt[3]{x}}\to\infty$

367. $\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^3} + \lim_{t\to 0^+}\int_t^2 \frac{dx}{x^3}$

As $x\to 0$ $\frac{1}{x^3}\to\infty$

check $\lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^3} = \lim_{t\to 0^-}\left[\frac{1}{-2x^2}\right]_{-1}^t = \lim_{t\to 0^-}\left[\frac{1}{-2t^2} - \left(\frac{1}{-2\cdot(-1)^2}\right)\right] = \lim_{t\to 0^-}\left[\frac{1}{-2t^2}+\frac{1}{2}\right]$

Since $\lim_{t\to 0^-}\frac{1}{-2t^2}$ diverges, then $\int_{-1}^0 \frac{dx}{x^3} = \lim_{t\to 0^-}\left[\frac{1}{-2t^2}+\frac{1}{2}\right]$ diverges.

So does $\int_{-1}^2 \frac{dx}{x^3}$

369. $\int_0^3 \frac{1}{x-1}dx = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = \lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} + \lim_{t\to 1^+}\int_t^3 \frac{dx}{x-1}$

As $x\to 1$, $\frac{1}{x-1}\to\infty$ since $\lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-}\left[\ln|x-1|\right]_0^t = \lim_{t\to 1^-}\left[\ln|t-1| - \ln|0-1|\right]$

and $\lim_{t\to 1^-}\ln|t-1| \to -\infty$, then $\lim_{t\to 1^-}\int_0^t \frac{dx}{x-1}$ diverges, so does $\int_0^3 \frac{dx}{x-1}$

371. $\displaystyle\int_3^5 \frac{5}{(x-4)^2}\,dx = \int_3^4 \frac{5}{(x-4)^2}\,dx + \int_4^5 \frac{5}{(x-4)^2}\,dx$

As $x \to 4$, $\frac{5}{(x-4)^2} \to \infty$ $= \lim_{t\to 4^-} \int_3^t \frac{5}{(x-4)^2}\,dx + \lim_{t\to 4^+} \int_t^5 \frac{5}{(x-4)^2}\,dx$

since $\lim_{t\to 4^-} \int_3^t \frac{5}{(x-4)^2}\,dx = \lim_{t\to 4^-} \left[\frac{-5}{x-4}\right]_3^t = \lim_{t\to 4^-} \left[\frac{-5}{t-4} - \left[\frac{-5}{3-4}\right]\right]$ and $\lim_{t\to 4^-} \frac{-5}{t-4}$ diverges.

Then $\lim_{t\to 4^-} \int_3^t \frac{5}{(x-4)^2}\,dx$ diverges, so does $\int_3^5 \frac{5}{(x-4)^2}\,dx$

Determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

373. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\displaystyle\int_1^\infty \frac{dx}{2\sqrt{x}}$.

since $\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} \le \frac{1}{\sqrt{x}+1}$ when $x \ge 1$

Then $\int_1^\infty \frac{dx}{2\sqrt{x}} \le \int_1^\infty \frac{dx}{\sqrt{x}+1}$

since $\int_1^\infty \frac{dx}{2\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{2\sqrt{x}} = \lim_{t\to\infty} \left[\sqrt{x}\right]_1^t = \lim_{t\to\infty} \left[\sqrt{t} - 1\right]$ diverges, then

so does $\int_1^\infty \frac{dx}{\sqrt{x}+1}$