Mat 1575 HW10 (§3.4)

Express the rational function as a sum or difference of two simpler rational expressions.

189. $\dfrac{1}{x^2(x-1)} = \dfrac{1}{x-1} + \dfrac{-1}{x} + \dfrac{A}{x^2} = \dfrac{x^2 - x(x-1) + A(x-1)}{x^2(x-1)}$

$\Rightarrow 1 = x^2 - x(x-1) + A(x-1)$

$\Rightarrow 1 = x^2 - x^2 + x + Ax - A \Rightarrow A = 1$ $\Rightarrow \dfrac{1}{x^2(x-1)} = \dfrac{1}{x-1} - \dfrac{1}{x} - \dfrac{1}{x^2}$

Use the method of partial fractions to evaluate each of the following integrals.

198. $\displaystyle\int \frac{dx}{x^3 - x} = \int \frac{dx}{x(x+1)(x-1)} = \int \frac{-1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\,dx$

$= -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$

205. $\displaystyle\int \frac{dx}{x^3 - 2x^2 - 4x + 8} = \int \frac{dx}{x(x-2)(x-4)} = \int \frac{\frac{1}{8}}{x} + \frac{\frac{-1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4}\,dx$

$x^3 - 2x^2 - 4x + 8$

$x^2(x-2) - 4x(x-2)$

$= (x-2)(x^2-4x) = (x-2)x(x-4)$

$= \frac{1}{8}\ln|x| - \frac{1}{4}\ln|x-2| + \frac{1}{8}\ln|x-4|$

206. $\displaystyle\int \frac{dx}{x^4 - 10x^2 + 9} = \int \frac{dx}{(x+1)(x-1)(x+3)(x-3)}$

$(x^2-1)(x^2-9)$

$= \int \left(\frac{1}{(-2)(-4)(-6)}\frac{}{x+3} + \frac{1}{(2)(-2)(-4)}\frac{}{x+1} + \frac{1}{4\cdot 2\cdot(-2)}\frac{}{x-1} + \frac{1}{6\cdot 4\cdot 2}\frac{}{x-3} \right) dx = \int \frac{\frac{1}{-48}}{x+3} + \frac{\frac{1}{16}}{x+1} + \frac{\frac{1}{-16}}{x-1} + \frac{\frac{1}{48}}{x-3}\,dx$

$= -\frac{1}{48}\ln|x+3| + \frac{1}{16}\ln|x+1| - \frac{1}{16}\ln|x-1| + \frac{1}{48}\ln|x-3| + C$

Evaluate the following integrals, which have irreducible quadratic factors.

207. $\displaystyle\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

$$= \int \frac{\frac{2}{4^2+2\cdot4+6}}{x-4} + \frac{Ax+B}{x^2+2x+6}\,dx$$

$$= \int \frac{\frac{1}{15}}{x-4} + \frac{Ax+B}{x^2+2x+6}\,dx$$

$$= \int \frac{\frac{1}{15}}{x-4} + \frac{-\frac{1}{15}x-\frac{6}{15}}{x^2+2x+6}\,dx$$

$$= \frac{1}{15}\ln|x-4| + \int \frac{-\frac{1}{15}x-\frac{1}{15}-\frac{5}{15}}{x^2+2x+6}\,dx$$

$$= \frac{1}{15}\ln|x-4| + \int \frac{-\frac{1}{15}(x+1)}{x^2+2x+6}\,dx + \int \frac{-\frac{1}{3}}{(x+1)^2+5}\,dx$$

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\cdot\frac{1}{2}\ln|x^2+2x+6| + \int \frac{-\frac{1}{3}\cdot\frac{1}{5}}{\frac{(x+1)^2}{5}+\frac{5}{5}}\,dx$$

$u = x^2+2x+6$, $du = (2x+2)\,dx$

$\Rightarrow \frac{du}{2} = (x+1)\,dx$

$\int \frac{dx}{x^2+1} = \tan^{-1}(x)+C$

$$= \frac{1}{15}\ln|x-4| - \frac{1}{30}\ln|x^2+2x+6| + \int \frac{-\frac{1}{15}}{\left(\frac{x+1}{\sqrt{5}}\right)^2+1}\,dx$$

$$= \frac{1}{15}\ln|x-4| - \frac{1}{30}\ln|x^2+2x+6| - \frac{1}{15}\cdot\sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$$

$$\frac{2}{(x-4)(x^2+2x+6)} = \frac{\frac{1}{15}(x^2+2x+6)+(Ax+B)(x-4)}{(x-4)(x^2+2x+6)}$$

$$\Rightarrow 2 = \frac{1}{15}x^2+\frac{2}{15}x+\frac{6}{15}+Ax^2+(B-4A)x-4B$$

$$\Rightarrow 0 = \underbrace{\left(\tfrac{1}{15}+A\right)}_{=0}x^2+\underbrace{\left(\tfrac{2}{15}+B-4A\right)}_{=0}x+\underbrace{\left(\tfrac{6}{15}-4B-2\right)}_{=0}$$

$\Rightarrow A = -\frac{1}{15}$, $B = \frac{-6}{15}$

209. $\displaystyle\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

$$= \int \frac{x^3+2x^2+4x^2+3x+6}{x^3+2x^2}\,dx$$

$$= \int \frac{x^3+2x^2}{x^3+2x^2} + \frac{4x^2+3x+6}{x^3+2x^2}\,dx = \int 1 + \frac{4x^2+3x+6}{x^2(x+2)}\,dx$$

$$= \int 1 + \frac{\frac{4(-2)^2+3(-2)+6}{(-2)^2}}{x+2} + \frac{A}{x} + \frac{\frac{4\cdot0^2+3\cdot0+6}{0+2}}{x^2}\,dx = \int 1 + \frac{4}{x+2} + \frac{A}{x} + \frac{3}{x^2}\,dx$$

$$\frac{4x^2+3x+6}{x^2(x+2)} = \frac{4}{x+2}+\frac{A}{x}+\frac{3}{x^2} = \frac{4x^2+Ax(x+2)+3(x+2)}{x^2(x+2)}$$

$4x^2+3x+6 = 4x^2+Ax(x+2)+3(x+2)$

move "$4x^2+3x+6$" to the other side $= 4x^2+Ax^2+2Ax+3x+6$

$\Rightarrow 0 = (4-4+A)x^2+(3+2A-3)x+(6-6)$

$\Rightarrow A=0.$

$$= \int 1 + \frac{4}{x+2} + \frac{3}{x^2}\,dx = x + 4\ln|x+2| + \frac{-3}{x} + C$$

210. $\displaystyle\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx = \int \frac{\frac{1}{(x^2+2x+2)^2}}{x-1} + \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}\,dx$

$$= \int \frac{\frac{1}{25}}{(x-1)} + \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}\,dx = I$$

$$\frac{x}{(x-1)(x^2+2x+2)^2} = \frac{\frac{1}{25}(x^2+2x+2)^2 + (Ax+B)(x-1)(x^2+2x+2) + (Cx+D)(x-1)}{(x-1)(x^2+2x+2)^2}$$

→ observation: $A = -\frac{1}{25}$

$$\Rightarrow x = \frac{1}{25}(x^2+2x+2)^2 + (Ax+B)(x-1)(x^2+2x+2) + (Cx+D)(x-1)$$

$$= \left(\tfrac{1}{25}+A\right)x^4 + \left(\tfrac{4}{25}+A+B\right)x^3 + \left(\tfrac{8}{25}+B+C\right)x^2 + \left(\tfrac{8}{25}-2A+D-C\right)x + \left(\tfrac{4}{25}-2B-D\right)$$

$$\Rightarrow 0 = \left(\tfrac{1}{25}+A\right)x^4 + \left(\tfrac{4}{25}+A+B\right)x^3 + \left(\tfrac{8}{25}+B+C\right)x^2 + \left(\tfrac{8}{25}-2A+D-C-1\right)x + \left(\tfrac{4}{25}-2B-D\right)$$

$\frac{1}{25}+A=0$, $A=-\frac{1}{25}$

$\frac{4}{25}-\frac{1}{25}+B=0$, $B=\frac{-3}{25}$

$\frac{8}{25}-\frac{3}{25}+C=0$, $C=\frac{-5}{25}=-\frac{1}{5}$

$\frac{4}{25}+\frac{6}{25}-D=0$, $D=\frac{10}{25}=\frac{2}{5}$

$$I = \int \frac{\frac{1}{25}}{(x-1)} + \frac{-\frac{1}{25}x-\frac{3}{25}}{x^2+2x+2} + \frac{-\frac{1}{5}x+\frac{2}{5}}{(x^2+2x+2)^2}\,dx$$

$$= \frac{1}{25}\ln|x-1| + \int\left(\frac{-\frac{1}{25}x-\frac{1}{25}}{x^2+2x+2} + \frac{-\frac{2}{25}}{x^2+2x+2}\right)dx + \int\left(\frac{-\frac{1}{5}x-\frac{1}{5}}{(x^2+2x+2)^2} + \frac{\frac{3}{5}}{(x^2+2x+2)^2}\right)dx$$

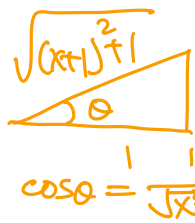
$$= \frac{1}{25}\ln|x-1| - \frac{1}{25}\cdot\frac{1}{2}\ln|x^2+2x+2| - \frac{2}{25}\tan^{-1}(x+1) - \frac{1}{5}\cdot\frac{-1}{2}\cdot\frac{1}{x^2+2x+2} + \frac{3}{10}\tan^{-1}(x+1) + \frac{6}{20}\frac{x+1}{x^2+2x+2} + C$$

$u = x^2+2x+2$, $du = (2x+2)\,dx$, $\frac{du}{2} = (x+1)\,dx$

$x^2+2x+2 = (x+1)^2+1$

$x+1 = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $(x+1)^2+1 = \sec^2\theta$

$\theta = \tan^{-1}(x+1)$, $\sin\theta = \frac{x+1}{\sqrt{x^2+2x+2}}$, $\cos\theta = \frac{1}{\sqrt{x^2+2x+2}}$



$$\int \frac{\frac{3}{5}\,dx}{(x^2+2x+2)^2} = \int \frac{\frac{3}{5}\,dx}{((x+1)^2+1)^2} = \int \frac{\frac{3}{5}\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int \frac{\frac{3}{5}}{\sec^2\theta}\,d\theta = \int \frac{3}{5}\cos^2\theta\,d\theta = \int \frac{3}{5}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{3}{10}\theta + \frac{3}{10}\cdot\frac{1}{2}\sin 2\theta + C = \frac{3}{10}\tan^{-1}(x+1) + \frac{3}{20}\,2\sin\theta\cos\theta + C$$

$$= \frac{3}{10}\tan^{-1}(x+1) + \frac{6}{20}\cdot\frac{x+1}{x^2+2x+2} + C$$

Use the method of partial fractions to evaluate the following integrals.

211. $\displaystyle\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

$= \displaystyle\int \frac{\frac{3\cdot 3+4}{(3)^2+4}}{3-x} + \frac{Ax+B}{x^2+4}\,dx$

$= \displaystyle\int \left(\frac{1}{3-x} + \frac{Ax+B}{x^2+4}\right)dx = \int \left(\frac{1}{3-x} + \frac{x}{x^2+4}\right)dx = \ln|x-3| + \frac{1}{2}\ln|x^2+4| + C$

$$\frac{3x+4}{(x^2+4)(3-x)} = \frac{1(x^2+4)+(3-x)(Ax+B)}{(x^2+4)(3-x)}$$

$\Rightarrow 3x+4 = (x^2+4)+(3-x)(Ax+B)$

- $x=0 \Rightarrow 4 = 4+(3-0)(0+B) \Rightarrow B=0$
- $x=1 \Rightarrow 7 = 5+(2)\cdot A \Rightarrow A=1$

212. $\displaystyle\int \frac{2}{(x+2)^2(2-x)}\,dx$

$= \displaystyle\int \frac{\frac{2}{(2+2)^2}}{2-x} + \frac{A}{x+2} + \frac{\frac{2}{(2-(-2))}}{(x+2)^2}\,dx$

$= \displaystyle\int \frac{\frac{1}{8}}{2-x} + \frac{A}{(x+2)} + \frac{\frac{1}{2}}{(x+2)^2}\,dx$

$= \displaystyle\int \frac{\frac{1}{8}}{2-x} + \frac{\frac{1}{8}}{(x+2)} + \frac{\frac{1}{2}}{(x+2)^2}\,dx$

$= -\frac{1}{8}\ln|2-x| + \frac{1}{8}\ln|x+2| - \frac{1}{2}\frac{1}{x+2} + C$

$$\frac{2}{(x+2)^2(2-x)} = \frac{\frac{1}{8}(x+2)^2 + A(x+2)(2-x) + \frac{1}{2}(2-x)}{(x+2)^2(2-x)}$$

- let $x=0 \Rightarrow 2 = \frac{1}{8}(0+2)^2 + A(0+2)(2-0) + \frac{1}{2}(2-0)$

$\Rightarrow 2 = \frac{1}{2} + 4A + 1 \Rightarrow A = \frac{1}{8}$

Use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

Let $u = e^x$, $du = e^x dx$

215. $\displaystyle\int \frac{e^x}{e^{2x}-e^x}\,dx$

$= \displaystyle\int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)} = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right)du$

$= -\ln|u| + \ln|u-1| + C = -\ln|e^x| + \ln|e^x-1| + C$

217. $\displaystyle\int \frac{\sin x}{\cos^2 x + \cos x - 6}\,dx$

Let $u = \cos x$, $du = -\sin x\,dx$

$$= \int \frac{-du}{u^2+u-6} = \int \frac{-du}{(u+3)(u-2)} = \int \frac{\frac{-1}{2+3}}{u-2} + \frac{\frac{-1}{3-2}}{u+3}\,du$$

$$= \int \frac{\frac{-1}{5}}{u-2} + \frac{\frac{1}{5}}{u+3}\,du = -\frac{1}{5}\ln|u-2| + \frac{1}{5}\ln|u+3| + C$$

$$= -\frac{1}{5}\ln|\cos(x)-2| + \frac{1}{5}\ln|\cos(x)+3| + C$$