Mat 1575 HW8 (§3.3)

Simplify the following expressions by writing each one using a single trigonometric function.

126. $4 - 4\sin^2\theta$ $= 4(1-\sin^2\theta) = 4\cos^2\theta$

128. $a^2 + a^2\tan^2\theta$ $= a^2(1+\tan^2\theta) = a^2\sec^2\theta$

Integrate using the method of trigonometric substitution. Express the final answer in terms of the variable.

135. $\int \frac{dx}{\sqrt{x^2-a^2}}$

$x = a\sec\theta \Rightarrow \sec\theta = \frac{x}{a}$

$dx = a\sec\theta\tan\theta\, d\theta$

$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{|a\tan\theta|}$ (*)

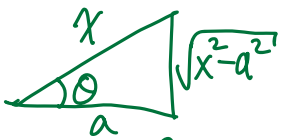
$x > a$ or $x < -a$

$\Rightarrow a\sec\theta > a$ or $a\sec\theta < -a$

$\Rightarrow \sec\theta > 1$ or $\sec\theta < -1$

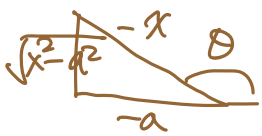
$\Rightarrow 0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$|\tan\theta| = \tan\theta$ $\quad$ $|\tan\theta| = -\tan\theta$



$\sec\theta = \frac{x}{a}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$



$\sec\theta = \frac{x}{a}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{-a}$

① If $0 \le \theta < \frac{\pi}{2}$, then $|a\tan\theta| = a\tan\theta$

(*) $= \int \frac{a\sec\theta\cdot\tan\theta}{a\tan\theta}\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$ $\rightarrow = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

② If $\frac{\pi}{2} < \theta \le \pi$, then $|a\tan\theta| = -a\tan\theta$

(*) $= \int \frac{a\sec\theta\tan\theta}{-a\tan\theta}\, d\theta$

$= \int -\sec\theta\, d\theta$

$= -\ln|\sec\theta + \tan\theta| + C$ $\rightarrow = -\ln\left|\frac{x}{a} - \frac{\sqrt{x^2-a^2}}{a}\right| + C$

137. $\int \frac{dx}{\sqrt{1+9x^2}}$

$\tan\theta = 3x$

$x = \frac{1}{3}\tan\theta$, $dx = \frac{1}{3}\sec^2\theta\, d\theta$

$\sqrt{1+9x^2} = \sqrt{1+9\left(\frac{1}{3}\tan\theta\right)^2}$

$= \sqrt{1+\tan^2\theta} = |\sec\theta| \overset{?}{=} \sec\theta$

$1+9x^2 \ge 0 \Rightarrow -\infty < x < \infty$

$\Rightarrow -\infty < \frac{1}{3}\tan\theta < \infty$

$\Rightarrow -\infty < \tan\theta < \infty$

$\Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow |\sec\theta| = \sec\theta$

$\tan\theta = \frac{3x}{1}$

$\sec\theta = \sqrt{1+9x^2}$

$= \int \frac{\frac{1}{3}\sec^2\theta}{\sec\theta}\, d\theta$

$= \int \frac{1}{3}\sec\theta\, d\theta = \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$

$= \frac{1}{3}\ln\left|\sqrt{1+9x^2} + 3x\right| + C$

$1-x^2 \geq 0 \Rightarrow -1 \leq x \leq 1$

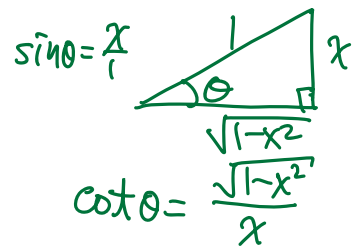
$-1 \leq \sin\theta \leq 1 \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Thus, $|\cos\theta| = \cos\theta$

139. $\int \frac{dx}{x^2\sqrt{1-x^2}}$

$x = \sin\theta$; $dx = \cos\theta\, d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$



$\cot\theta = \frac{\sqrt{1-x^2}}{x}$

$= \int \frac{\cos\theta\, d\theta}{\sin^2\theta\,|\cos\theta|} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta \cdot \cos\theta} = \int \frac{d\theta}{\sin^2\theta}$

$= \int \csc^2\theta\, d\theta = -\cot\theta + C = -\frac{\sqrt{1-x^2}}{x} + C$

141. $\int \sqrt{x^2+9}\,dx$

$x = 3\tan\theta$; $dx = 3\sec^2\theta\, d\theta$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3|\sec\theta|$

$x^2+9 \geq 0 \Rightarrow -\infty < x < \infty$

$\Rightarrow -\infty < 3\tan\theta < \infty$

$\Rightarrow -\infty < \tan\theta < \infty$

$\Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow |\sec\theta| = \sec\theta$

$\tan\theta = \frac{x}{3}$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$= \int 3|\sec\theta|\,d\theta = \int 3\sec\theta\, d\theta$

$= 3\ln\left|\sec\theta + \tan\theta\right| + C$

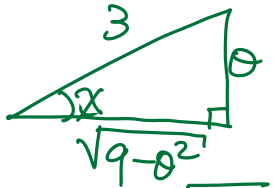
$= 3\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$

143. $\int \frac{\theta^3\, d\theta}{\sqrt{9-\theta^2}}$

$\theta = 3\sin x$; $d\theta = 3\cos(x)\,dx$

$\sqrt{9-\theta^2} = \sqrt{9-9\sin^2 x} = 3|\cos x| = 3\cos x$ (x) similar statement ....

$\sin x = \frac{\theta}{3}$



$\cos x = \frac{\sqrt{9-\theta^2}}{3}$

$= \int \frac{(3\sin x)^3}{3\cos x} \cdot 3\cos x\, dx = \int (3\sin x)^3\, dx$

$= \int 27\sin^2(x)\sin(x)\,dx = \int 27\left(1-\cos^2(x)\right)\sin(x)\,dx$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$= \int -27(1-u^2)\,du = \int 27u^2 - 27\,du$

$= 9u^3 - 27u + C = 9\cos^3(x) - 27\cos(x) + C$

$= 9\left(\frac{\sqrt{9-\theta^2}}{3}\right)^3 - 27\frac{\sqrt{9-\theta^2}}{3} + C$

$= \frac{\left(\sqrt{9-\theta^2}\right)^3}{3} - 9\sqrt{9-\theta^2} + C$

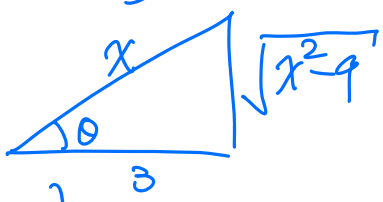
147. $\int \frac{dx}{(x^2-9)^{3/2}}$

$x=3\sec\theta$, $dx=3\sec\theta\tan\theta\,d\theta$

$(x^2-9)^{\frac{3}{2}}=\left(\sqrt{x^2-9}\right)^3=\left(\sqrt{(3\sec\theta)^2-9}\right)^3$

$=(3|\tan\theta|)^3$

$\sec\theta=\frac{x}{3}$



$\sin\theta=\frac{\sqrt{x^2-9}}{x}$

① If $0\le\theta<\frac{\pi}{2}$, $|\tan\theta|=\tan\theta$, then

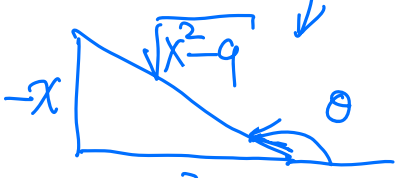
$\int\frac{dx}{(x^2-9)^{\frac{3}{2}}}=\int\frac{3\sec\theta\cdot\tan\theta\,d\theta}{(3\tan\theta)^3}=\int\frac{\sec\theta}{9\tan^2\theta}d\theta$

$=\int\frac{\frac{1}{\cos\theta}}{9\frac{\sin^2\theta}{\cos^2\theta}}d\theta=\int\frac{1}{9}\frac{\cos\theta}{\sin^2\theta}d\theta=\int\frac{1}{9}\frac{du}{u^2}$

$u=\sin\theta$, $du=\cos\theta\,d\theta$

$=\frac{-1}{9}\frac{1}{u}+C=-\frac{1}{9}\frac{1}{\sin\theta}+C=-\frac{1}{9}\frac{x}{\sqrt{x^2-9}}+C$

$\sec\theta=\frac{x}{3}$



$\sin\theta=\frac{-x}{\sqrt{x^2-9}}$

② If $\frac{\pi}{2}<\theta\le\pi$, $|\tan\theta|=-\tan\theta$, then

$\int\frac{dx}{(x^2-9)^{\frac{3}{2}}}=\int\frac{3\sec\theta\cdot\tan\theta}{(-3\tan\theta)^3}=\int-\frac{\sec\theta}{9\tan^2\theta}d\theta$

$=\cdots=\frac{1}{9}\frac{1}{\sin\theta}+C=-\frac{1}{9}\frac{x}{\sqrt{x^2-9}}+C$

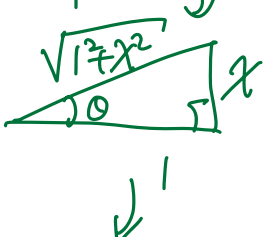
151. $\int\frac{dx}{x^2\sqrt{x^2+1}}$

$x=\tan\theta$, $dx=\sec^2\theta\,d\theta$

$\sqrt{x^2+1}=\sqrt{\tan^2\theta+1}=|\sec\theta|$

$=\sec\theta$

$\tan\theta=\frac{x}{1}$



$\sin\theta=\frac{x}{\sqrt{1+x^2}}$

$=\int\frac{\sec^2\theta}{\tan^2\theta\cdot\sec\theta}d\theta$

$=\int\frac{\sec\theta}{\tan^2\theta}d\theta=\int\frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos\theta}}d\theta=\int\frac{\cos\theta}{\sin^2\theta}d\theta$

$u=\sin\theta$, $du=\cos\theta\,d\theta$

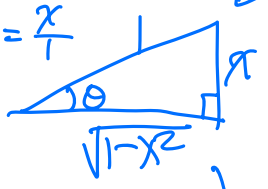
$=\int\frac{du}{u^2}=-\frac{1}{u}+C=-\frac{1}{\sin\theta}+C=-\frac{\sqrt{1+x^2}}{x}+C$

153. $\int(1-x^2)^{3/2}dx$

$x=\sin\theta$, $dx=\cos\theta\,d\theta$

$(1-x^2)^{\frac{3}{2}}=\left(\sqrt{(1-\sin^2\theta)}\right)^3=|\cos\theta|^3=\cdots=\cos^3\theta$

$\sin\theta=\frac{x}{1}$



$=\int\cos^3\theta\cdot\cos\theta\,d\theta=\int\cos^4\theta\,d\theta=\int\cos^2\theta\cdot\cos^2\theta\,d\theta$

$$= \int \left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right) d\theta = \int \frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{\cos^2(2\theta)}{4}\, d\theta$$

$\cos\theta = \sqrt{1-x^2}$

$\theta = \sin^{-1}(x)$

$$= \int \frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{1}{4}\cdot\frac{1+\cos(4\theta)}{2}\, d\theta$$

$$= \int \frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{1}{8} + \frac{\cos(4\theta)}{8} d\theta = \int \frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{8}$$

$$= \frac{3}{8}\theta + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} + C$$

$$= \frac{3}{8}\theta + \frac{1}{4}\cdot 2\sin\theta\cdot\cos\theta + \frac{1}{32} 2\sin 2\theta\cos 2\theta + C$$

$$= \frac{3}{8}\theta + \frac{\sin\theta\cos\theta}{2} + \frac{(2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta)}{16} + C$$

$$= \frac{3}{8}\sin^{-1}(x) + \frac{1}{2}\cdot x\cdot\sqrt{1-x^2} + \frac{1}{16}(2x\sqrt{1-x^2})(1-x^2-x^2) + C$$

$$= \frac{3}{8}\sin^{-1}(x) + \frac{1}{2}x\cdot\sqrt{1-x^2} + \frac{1}{8}(2x\sqrt{1-x^2})(1-2x^2) + C.$$