MAT1575, Classwork17, Fall2026

5.6 Ratio Tests and Root Tests (p.449-456)

1. *Theorem*: **Ratio Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms and $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. *Then* $\sum_{n=1}^{\infty} a_n$ $\begin{cases} \text{converges absolutely} & \text{if } 0 \le \rho < 1 \\ \text{inconclusive} & \text{if } \rho = 1 \\ \text{diverges} & \text{if } \rho > 1 \text{ or } \rho \to \infty \end{cases}$

2. For each given series, use the ratio test determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 \Rightarrow$ By ratio test, $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges

$a_n = \frac{2^n}{n!}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

(b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{1}{n+1} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$

$a_n = \frac{n^n}{n!}$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by ratio test

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ $a_{n+1} = \frac{(-1)^{n+1}((n+1)!)^2}{(2(n+1)!)}$, $\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{2n!}{(-1)^n (n!)^2} = \frac{(-1)(n+1)(n+1)}{(2n+2)(2n+1)}$

$a_n = \frac{(-1)^n (n!)^2}{(2n)!}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)(n+1)(n+1)}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, Then, by ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ converges.

3. *Theorem*: **Root Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms and $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. *Then* $\sum_{n=1}^{\infty} a_n$ $\begin{cases} \text{converges absolutely} & \text{if } 0 \le \rho < 1 \\ \text{inconclusive} & \text{if } \rho = 1 \\ \text{diverges} & \text{if } \rho > 1 \text{ or } \rho \to \infty \end{cases}$

4. For each given series, use the root test determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$ $a_n = \frac{(n^2+3n)^n}{(4n^2+5)^n}$ $\sqrt[n]{|a_n|} = \frac{n^2+3n}{4n^2+5}$ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}$

$|a_n|^{\frac{1}{n}}$ by Root test, $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

5. Evaluate $\lim_{n\to\infty} \sqrt[n]{n}$

6. For each given series, use both the root and ratio test determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{n}$

$(b) \sum_{n=1}^{\infty} \frac{1}{n^2}$

7. Summary of Convergence Tests:

Test of Series	*Convergence Cond.*	*Divergence Cond.*	*Inconclusive*
Divergence Test: $\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	$\lim_{n\to\infty} a_n = 0$
Integral Test: $\sum_{n=1}^{\infty} a_n$, $f(n) = a_n > 0$, f continuous, f decreasing	$\int_1^{\infty} f(x)\,dx \; converges$	$\int_1^{\infty} f(x)\,dx \; diverges$	
p Series Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Comparison Test: $\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n \; and \; \sum_{n=1}^{\infty} b_n \; conv.$	$0 \leq b_n \leq a_n \; and \; \sum_{n=1}^{\infty} b_n \; div.$	
Limit Comparison Test: $\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0 \;\; and \;\; \sum_{n=1}^{\infty} b_n \; conv.$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0 \;\; and \;\; \sum_{n=1}^{\infty} b_n \; div.$	
Alternating Series Test: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		
Ratio Test: $\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$
Root Test: $\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$
Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r \rvert < 1$ and Sum $= \frac{a}{1-r}$	$\lvert r \rvert \geq 1$	
Telescoping Series: $\sum_{n=1}^{\infty} (a_n - a_{n+1})$	$\lim_{n\to\infty} a_n = L$ and Sum $= a_1 - L$		