MAT1575, Classwork16, Fall2026

5.4 Comparison Tests II (p.426-433)

3. *Theorem*: **Limit Comparison Test**

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. ($a_n < b_n$, b_n converges, a_n converges)

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. ($0 < b_n < a_n$, b_n diverges, a_n diverges)

4. For each given series, use the limit comparison test to determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ Compare this series with $\sum \frac{1}{n^2}$ (b_n)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$ (fixed number)

By limit Comparison Test, since $\sum \frac{1}{n^2}$ converg (by p-series), then $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ (a_n) Compare this series with $\sum \frac{1}{\sqrt{n}}$ (diverges, by p-series) (b_n)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = 1$

Then, by limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ Compare this series with $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ (Converges since $r = \frac{2}{3} < 1$)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n+1}{3^n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{2^n+1}{2^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} 1+\frac{1}{2^n} = 1$

Then, by limit Comparison test, $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges

(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ (Conv.?) $b_n = \frac{1}{n^2}$ $\frac{a_n}{b_n} = \frac{\ln(n)}{n^2} \cdot \frac{n^2}{1} = \ln(n)$, $\lim_{n\to\infty} \ln(n) \to \infty$, it fails, inconclusive.

(div.) $b_n = \frac{1}{n}$ $\frac{a_n}{b_n} = \frac{\ln(n)}{n^2} \cdot \frac{n}{1} = \frac{\ln(n)}{n}$, $\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty} \frac{1}{1} = \frac{0}{1} = 0$, it fails, inconclusive

$b_n = \frac{1}{n^{\frac{3}{2}}}$, $\frac{a_n}{b_n} = \frac{\ln(n)}{n^2} \cdot \frac{n^{\frac{3}{2}}}{1} = \frac{\ln(n)}{n^{\frac{1}{2}}}$, $\lim_{n\to\infty} \frac{\ln(n)}{n^{\frac{1}{2}}} \overset{L'}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty} \frac{2n^{\frac{1}{2}}}{n} = \lim_{n\to\infty} \frac{2}{n^{\frac{1}{2}}} = 0$ ($\frac{1}{2}n^{-\frac{1}{2}} = \frac{1}{2n^{\frac{1}{2}}}$)

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ conv. (by p-series), then $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges by limit Comparison test.

5.5 Alternating Series (p.426-433)

1. *Definition*: **Alternating Series**.

Any series whose terms alternate between positive and negative is called an alternating series and it can be written in the form with $b_n > 0$ for all positive integers n:

$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$ or $\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 - \cdots$

2. *Examples.*

$-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ → Geometric Series : common ratio $|r|<1$

(a) $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$ is an alternating series and converges to $\frac{a}{1-r}=\frac{-\frac{1}{2}}{1-(-\frac{1}{2})}=(-\frac{1}{2})\cdot\frac{2}{3}=-\frac{1}{3}$ (a: 1st term, $r=-\frac{1}{2}$)

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is an alternating series (converges?) Yes, by Alternating series Test

$=\left|-\frac{1}{2}+\frac{1}{3}-\cdots\right.$ $b_n=\frac{1}{n}$ ① $0<b_{n+1}\le b_n$ (√) ② $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{n}=0$ (√)

3. *Theorem*: **Alternating Series Test**

An alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^n b_n$ converges if (i) $0<b_{n+1}\le b_n$ for all n ≥ 1 and (ii) $\lim_{n\to\infty} b_n=0$

4. For each given alternating series, determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$, $b_n=\frac{1}{n^2}$, $b_{n+1}=\frac{1}{(n+1)^2}\Rightarrow 0<b_{n+1}\le b_n$ and $\lim_{n\to\infty} b_n=0$

By Alternating Series Test, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ converges.

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$, $b_n=\frac{n}{n+1}$, $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{n}{n+1}=1$, by divergence Test,

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$ diverges.

5. *Definition*: **Absolute Convergence and Conditional Convergence**

A series $\sum_{n=1}^{\infty} a_n$ exhibits absolute Convergence if $\sum_{n=1}^{\infty}|a_n|$ converges,

A series $\sum_{n=1}^{\infty} a_n$ exhibits Conditional Convergence if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges

6. Example of Conditional Convergence: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series Test

by alternating Series Test, it converges

7. What is the difference between absolute and conditional convergence?

When the terms are rearranged in a conditional converges series, the series will become divergent.

8. For each given series, determine whether the series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+1}$ $b_n=\frac{1}{3n+1}>b_{n+1}=\frac{1}{3(n+1)+1}=\frac{1}{3n+4}>0$, $\lim_{n\to\infty}\frac{1}{3n+1}=0$, it converges by Alternating Series Test.

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+1}\right|=\sum_{n=1}^{\infty}\frac{1}{3n+1}$ $\sum_{n=1}^{\infty}\frac{1}{n}$ (by p-series) diverges.

by limit comparison Test $\lim_{n\to\infty}\frac{\frac{1}{3n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{3n+1}=\frac{1}{3}\neq 0$ $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+1}\right|$ diverges

(b) $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$ $0\le\sum_{n=1}^{\infty}\left|\frac{\cos(n)}{n^2}\right|\le\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-series Test ⇒ It's conditional convergent.

$|\cos(n)|\le 1$, Therefore $\sum_{n=1}^{\infty}\left|\frac{\cos(n)}{n^2}\right|$ converges

which implies $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$ converges Absolutely