MAT1575, Classwork15, Fall2026

5.3 The Divergence and the Integral Tests (p.413-422)

1. *Theorem*: **Divergence Test**.

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

$a_n \to 0$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

2. If $\lim_{n\to\infty} a_n = 0$, then does $\sum_{n=1}^{\infty} a_n$ diverge? NOT necessarily, inconclusive

3. Show each given series diverges by using Divergence Test. If the test fails, indicate that the it is *inconclusive*.

(a) $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ $a_n = \frac{n}{3n-1}$

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0$, then, by divergence test $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $a_n = \frac{1}{n^3}$ $\lim_{n\to\infty} a_n = 0$ inconclusive

(c) $\sum_{n=1}^{\infty} e^{1/n^2}$ $a_n = e^{\frac{1}{n^2}}$

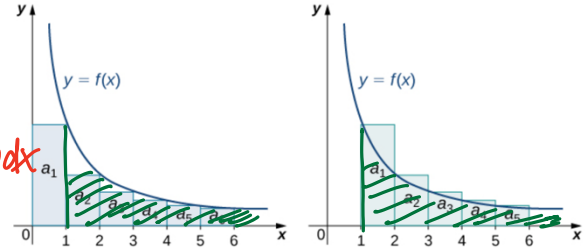
since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{1}{n^2}} = e^0 = 1 \neq 0$,

by divergence test, $\sum_{n=1}^{\infty} e^{\frac{1}{n^2}}$ diverges

4. *Theorem*: **Integral Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$. Suppose there exists a function f and a positive integer N such that (i) $\boldsymbol{f}$ **is continuous**, (ii) $\boldsymbol{f}$ **is decresing**, and (iii) $\boldsymbol{f(n) = a_n}$ **for all integers** $\boldsymbol{n \geq N}$.

Then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.



CAUTION: This Theorem doesn't imply $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$

5. Show the convergence of each given series by using Integral Test.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $f(x) = \frac{1}{x^3}$ ① continuous, ② decreasing ③ $f(n) = a_n$

since $\int_1^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^3} = \lim_{t\to\infty} \left[\frac{-1}{2x^2}\right]_1^t = \lim_{t\to\infty} \left[\frac{-1}{2t^2} - \left(\frac{-1}{2(1)^2}\right)\right] = \frac{1}{2}$ converges,

then $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by integral test.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ $f(x) = \frac{1}{\sqrt{2x-1}}$ ① It's continuous, ② decreasing ③ $f(n) = a_n$

$\int_1^{\infty} \frac{dx}{\sqrt{2x-1}} = \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{2x-1}} = \lim_{t\to\infty} \left[\sqrt{2x-1}\right]_1^t = \lim_{t\to\infty} \left[\sqrt{2t-1} - (\sqrt{2-1})\right]$ diverges

then, by integral test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{1}{n}$ $f(x) = \frac{1}{x}$, ① continuous, ② decreasing ③ $f(n) = a_n$

$\int_1^{\infty} \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \left[\ln|x|\right]_1^t = \lim_{t\to\infty} \left[\ln|t| - \ln|1|\right]$ diverges

Then, by integral test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

6. *Definition*: *p*-Series

For any real number p, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a *p* series

7. **The convergence of *p*-Series**: $\frac{1}{x^{-2}} = \frac{1}{\frac{1}{x^2}} = x^2 \to \infty,\ x \to \infty$

If $p < 0$, then $\frac{1}{n^p} \to \infty$ as $n \to \infty$. By Divergence Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

If $p = 0$, then $\frac{1}{n^p} \to 1$ as $n \to \infty$. By Divergence Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

If $p > 0$, then $f(x) = \frac{1}{x^p}$ is a continuous, decreasing, $f(n) = \frac{1}{n^p}$, positive. Therefore, for $p > 0$, we can use the integral test.

- If $p = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test.
- If $p \neq 1$, $\int_1^{\infty} \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p} = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \left(\frac{1}{-p+1}\right)\right] = \begin{cases} \frac{1}{1-p} & \text{converges } p>1 \\ \text{diverges} & 0<p<1 \ (1-p>0) \end{cases}$

($\frac{1}{(-p+1)\cdot t^{p-1}}$)

In conclusion, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges}, & p>1 \\ \text{diverges} & p \le 1 \ (p<0,\ p=1,\ 0<p<1) \end{cases}$

8. Show the convergence of each given series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ $p = 4 > 1$, By p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p = \frac{2}{3} < 1$ By p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ diverges

5.4 Comparison Tests I (p.426-433)

1. *Theorem*: **Comparison Test** (Direct) $0 \le \sum a_n \le \sum b_n$ $\sum_{n=1}^{\infty} a_n$ conv? div?

Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. Then we have

(*i*) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(*ii*) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. For each given series, use the comparison test to determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ Compare this series with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\frac{1}{n^3+3n+1} < \frac{1}{n^3}$

since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (by p-series test), then, by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ conv.

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ Compare this series with $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ by Geometric series, converges

$\frac{1}{2^n+1} < \frac{1}{2^n}$, then $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges. $|r| = \left|\frac{1}{2}\right| < 1$ $\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

(c) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ Compare this series with $\sum_{n=2}^{\infty} \frac{1}{n}$ since $\frac{1}{\ln(n)} > \frac{1}{n}$ and, by p-series test, $\sum \frac{1}{n}$ diverges

then, by comparison test, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ Compare this series: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\frac{1}{n^2-1} > \frac{1}{n^2}$ inconclusive