MAT1575, Classwork14, Fall2026

5.2 Infinite Series (p.394-408)

1. *Definition*: **Infinite Series** and **Partial Sum**.

A series which sums up infinite terms is called An Infinite Series : $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+a_4+\cdots$

For each positive integer k, we define kth partial sum: $S_k = a_1+a_2+a_3+a_4+\cdots+a_{k-1}+a_k$, and $a_k = S_k - S_{k-1}$ ($S_{k-1} = a_1+a_2+a_3+a_4+\cdots+a_{k-1}$)

The partial sums form a sequence $\{S_k\}$. If $\{S_k\}$ converges to a real number S, this infinite series converges and we called it the convergence of a series to S (the sum of the series), and write $\sum_{n=1}^{\infty} a_n = S$

If the sequence of partial sum diverges, we have the divergence of a series.

2. *Example*. For each series, use the sequence of partial sums to determine if the series converges or diverges.

$(a)\ \sum_{n=1}^{\infty} \frac{n}{n+1}$ $\quad a_n = \frac{n}{n+1}$

$S_1 = a_1 = \frac{1}{1+1} = \frac{1}{2}$

$S_2 = a_1+a_2 = \frac{1}{2}+\frac{2}{3} > \frac{1}{2}+\frac{1}{2} = 2(\frac{1}{2})$

$S_3 = a_1+a_2+a_3 = \frac{1}{2}+\frac{2}{3}+\frac{3}{4} > \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 3(\frac{1}{2})$

$S_4 = a_1+a_2+a_3+a_4 = \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5} > \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 4(\frac{1}{2})$

$S_n > n(\frac{1}{2}) = \frac{n}{2}$

since $n\to\infty$, $\frac{n}{2}\to\infty$

then $S_n \to \infty$ (S_n diverges)

$(b)\ \sum_{n=1}^{\infty} (-1)^n$ $\quad a_n = (-1)^n$

$S_1 = a_1 = (-1)^1 = -1$

$S_2 = a_1+a_2 = -1+(-1)^2 = -1+1 = 0$

$S_3 = a_1+a_2+a_3 = S_2+(-1)^3 = -1$

$S_4 = 0$

since $\{S_n\}$ diverges, then $\sum_{n=1}^{\infty} (-1)^n$ diverges

$(c)\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $\quad a_n = \frac{1}{n(n+1)}$

$S_1 = a_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$

$S_2 = a_1+a_2 = \frac{1}{2}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$S_3 = a_1+a_2+a_3 = \frac{2}{3}+\frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$S_4 = a_1+a_2+a_3+a_4 = \frac{3}{4}+\frac{1}{20} = \frac{16}{20} = \frac{4}{5}$

since $S_n = \frac{n}{n+1}$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$

Then $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

3. The alternative way to solve 2(c).

$\frac{1}{n(n+1)} = \frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$

$\frac{1}{n} - \frac{1}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$ (the very last term goes to zero)

$= 1 + 0 = 1$

4. *Definition*: **Telescoping Series**

A Telescoping Series is a series in which most of the terms cancel in each of the partial sums, leaving only the first terms and some of the last terms

5. Show the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. $a_n = \frac{1}{n}$

$S_1 = a_1 = 1$

$S_2 = a_1+a_2 = 1+\frac{1}{2}$

$S_4 = a_1+a_2+a_3+a_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+2(\frac{1}{2})$

$S_8 = a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

$> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+3(\frac{1}{2})$ ($\frac{1}{8} > \frac{1}{8}$)

$(j \ge 0)$

$S_{2^j} > 1+j(\frac{1}{2})$, as $j\to\infty$, $1+j(\frac{1}{2})\to\infty$

so S_{2^j} diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

6. **Algebraic Properties of Convergent Series**.

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series and c be any real number. Then the following algerbic properties hold.

(a) Sum Rule: The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(b) Difference Rule: The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

(c) Constant Multiple Rule: The series $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c\cdot\sum_{n=1}^{\infty} a_n$

7. Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right] = \sum_{n=1}^{\infty}\frac{3}{n(n+1)} + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2} = 3 + 4 = 7$

$\sum_{n=1}^{\infty}\frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 3\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 3\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \cdots\right) = 3$

$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2} = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 2\cdot 2 = 4$ common ratio. $r = \frac{1}{2}$

$a_n = \left(\frac{1}{2}\right)^{n-2}$, $a_1 = \left(\frac{1}{2}\right)^{-1} = (2^{-1})^{-1} = 2$, $a_2 = \left(\frac{1}{2}\right)^{0} = 1$, $a_3 = \left(\frac{1}{2}\right)^{3-2} = \frac{1}{2}$

8. The Convergence of **Geometric Series** with the first term a and the common ratio r

A geometric series is a series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a_1 + a\cdot r + a\cdot r^2 + a\cdot r^3 + ar^4 + \cdots$

If $|r| < 1$, the series converges, and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a_1}{1-r}$. If $|r| \geq 1$, the series diverges

9. Determine if each given series converges. If yes, find its sum.

(a) $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}} = \sum_{n=1}^{\infty}\frac{(-3)^2\cdot(-3)^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty}(-3)^2\cdot\frac{(-3)^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty} 9\cdot\left(\frac{-3}{4}\right)^{n-1} = \frac{a_1}{1-r} = \frac{9}{1-(-\frac{3}{4})} = \frac{9}{\frac{7}{4}} = 9\cdot\frac{4}{7} = \frac{36}{7}$

$a_1 = 9$, $r = -\frac{3}{4}$

$\left(\sum_{n=1}^{\infty} a\cdot r^{n-1}\right)$

(b) $\sum_{n=1}^{\infty} e^{2n} = \sum_{n=1}^{\infty} e^2\cdot e^{2n-2} = \sum_{n=1}^{\infty} e^2(e^2)^{n-1}$ since $r = e^2 > 1$, then

$a = e^2$, $r = e^2 > 1$

$e \approx 2.71\cdots$

this geometric series diverges.