MAT1575, Classwork13, Fall2026

$a_1=5,\ a_n=a_{n-1}+2$

5.1 Sequences (p.374-390) $\{a_n\}: a_n=3+2n,\ a_1=3+2=5,\ a_2=7,\ a_3=9,\ \ldots$ terms

1. *Definition*: **Terminology of Sequences**

Infinite sequence $\{a_n\}$: an ordered list of numbers with index value n and they are

Explicit formulas: $a_n=f(n)$ for some function f(n) defined over positive integers.

Recurrence Relation: one or more terms are given, and subsequent terms are defined in terms in the sequence

2. *Examples*.

(a) **Arithmetic Sequence**: Given $a_1=3, a_n=a_{n-1}+4$ for $n\geq 2$. Find the explicit formula for a_n.

$a_1=3,\ a_2=a_1+4=7,\ a_3=a_2+4=11,\ a_4=a_3+4=15,\ \ldots\ a_5=a_1+4\cdot 4$

$=a_1+4+4$ $=a_1+4+4+4$ $=a_1+3\cdot 4$ $\Rightarrow a_n=a_1+(n-1)\cdot 4$

In general, $a_1=3$ is the first term, $d=a_n-a_{n-1}=$ common difference $a_n=3+(n-1)\cdot 4$

(b) **Geometric Sequence**: Given $a_1=2, a_n=-\frac{1}{3}\cdot a_{n-1}$ for $n\geq 2$. Find the explicit formula for a_n.

$a_1=2,\ a_2=(-\frac{1}{3})a_1=-\frac{2}{3},\ a_3=-\frac{1}{3}a_2=\frac{2}{9},\ a_4=-\frac{1}{3}a_3=\frac{2}{9}\cdot(-\frac{1}{3})=-\frac{2}{27}$

$=-\frac{1}{3}\cdot(-\frac{1}{3}a_1)=(-\frac{1}{3})^2a_1$ $=-\frac{1}{3}(-\frac{1}{3})^2a_1=(-\frac{1}{3})^3a_1$ $a_n=(-\frac{1}{3})^{n-1}a_1=(-\frac{1}{3})^{n-1}\cdot 2$ $=2\cdot(-\frac{1}{3})^{n-1}$

In general, $a_1=2$ is the first term, common ratio $r=\frac{a_n}{a_{n-1}}=-\frac{1}{3}$

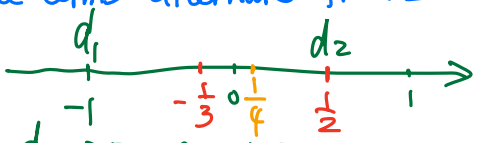
3. *Examples*: Find the Limit of each Sequence when $n\to\infty$.

(a) $\{a_n\}=\{3n+1\}$: $a_n=3n+1$, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}3n+1\to\infty$ diverges

(b) $\{b_n\}=\left\{1-\left(\frac{1}{2}\right)^n\right\}$: $b_1=1-(\frac{1}{2})^n$, $\lim_{n\to\infty}b^n=\lim_{n\to\infty}\left(1-(\frac{1}{2})^n\right)=1$

(c) $\{c_n\}=\{(-1)^n\}$: $c_n=(-1)^n$, The terms alternate from 1 to -1 but do not approach to one fixed number $\Rightarrow$ diverges.

(d) $\{d_n\}=\left\{\frac{(-1)^n}{n}\right\}$: $d_n=\frac{(-1)^n}{n}$,



as $n\to\infty$, $d_n\to 0$ converges

4. *Definition*: **Convergent and Divergent Sequence**.

Given a sequence $\{a_n\}$. If the term a_n become arbitrarily close to a finite number L as $n\to\infty$, we say $\{a_n\}$ is a convergent sequence and L is the limit of the sequence: $\lim_{n\to\infty}a_n=L$

If $\{a_n\}$ is not convergent, we say it is a divergent sequence

5. **Algebraic Limit Laws**. Given sequences $\{a_n\},\{b_n\}$ and a real number c. If there exist finite numbers A and B such that $\lim_{n\to\infty}a_n=A$, $\lim_{n\to\infty}b_n=B$.

(a) $\lim_{n\to\infty}c=c$ (b) $\lim_{n\to\infty}ca_n=c\cdot\lim_{n\to\infty}a_n=cA$.

(c) $\lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n=A\pm B$

(d) $\lim_{n\to\infty}(a_n\cdot b_n)=(\lim_{n\to\infty}a_n)(\lim_{n\to\infty}b_n)=A\cdot B$

(e) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}=\frac{A}{B}$

6. *Examples*: For each given sequence, determine whether or not the sequence converge and find its limit.

(a) $\left\{\frac{3n^4-7n^2+5}{6-4n^4}\right\}$ $\lim_{n\to\infty}\frac{3n^4-7n^2+5}{-4n^4+6} \overset{L'}{=} \lim_{n\to\infty}\frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{-4+\frac{6}{n^4}} = \frac{3}{-4}$, converges

(b) $\left\{\frac{2^n}{n^2}\right\}$ $\lim_{n\to\infty}\frac{2^n}{n^2} \overset{(\frac{\infty}{\infty})}{=} \lim_{n\to\infty}\frac{2^n\cdot\ln 2}{2n} \overset{(\frac{\infty}{\infty})}{=} \lim_{n\to\infty}\frac{2^n\cdot(\ln 2)^2}{2} = \lim_{n\to\infty} 2^{n-1}\cdot(\ln 2)^2 \to \infty$ diverges

(c) $\left\{\left(1+\frac{4}{n}\right)^n\right\}$ $\lim_{n\to\infty}(1+\frac{4}{n})^n = ?$ e^4

let $y=\lim_{n\to\infty}(1+\frac{4}{n})^n$, $\ln(y)=\lim_{n\to\infty} n\cdot\ln(1+\frac{4}{n}) = \infty\cdot 0 \to \frac{\infty}{\infty}$ (L) $= \lim_{n\to\infty}\frac{\frac{1}{n+4}-\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{-4}{n(n+4)}\cdot\frac{1}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{4n^2}{n(n+4)} = \lim_{n\to\infty}\frac{4n^2}{n^2+4n} = 4$

$\lim_{n\to\infty}\frac{\ln(\frac{n+4}{n})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\ln(n+4)-\ln(n)}{\frac{1}{n}}$

$\ln(y)=4 \Rightarrow y=e^4$

$\log_e y = 4$

7. **Squeeze Theorem** for Sequences.

Consider sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N and a finite number L such that $a_n \le b_n \le c_n$ for all $n \ge N$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

8. Use the Squeeze Theorem to find the limit of each given sequence:

(a) $\left\{\frac{\cos(n)}{n^2}\right\}$ $\frac{-1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$

$0 = \lim_{n\to\infty}\frac{-1}{n^2} \le \lim_{n\to\infty}\frac{\cos(n)}{n^2} \le \lim_{n\to\infty}\frac{1}{n^2} = 0$

(b) $\left\{\left(-\frac{1}{2}\right)^n\right\}$ $-\left(\frac{1}{2}\right)^n \le \left(-\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^n$

$0 = \lim_{n\to\infty} -\left(\frac{1}{2}\right)^n \le \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n \le \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$

This implies $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$

9. *Definition*: **Convergent and Bounded Sequences**. Given a sequence $\{a_n\}$ and a real number M.

(a) Bounded above: $\{a_n\}$ is bounded above if $a_n \le M$ for all positive integer n

(b) Bounded below: $\{a_n\}$ is bounded below if $m \le a_n$ for all positive integer n

(c) $\{a_n\}$ is a bounded sequence if it is bounded above and bounded below.

(d) If $\{a_n\}$ is not a bounded sequence, it is an unbounded sequence.

(e) If $\{a_n\}$ converges, then it is bounded.

10. *Definition*: **Monotone Sequence**. Given a sequence $\{a_n\}$.

(a) $\{a_n\}$ is increasing for all $n \ge n_0$ if $a_n \le a_{n+1}$ for all $n \ge n_0$

(b) $\{a_n\}$ is decreasing for all $n \ge n_0$ if $a_n \ge a_{n+1}$ for all $n \ge n_0$

(c) $\{a_n\}$ is a monotone sequence for all $n \ge n_0$ if it is either increase or decrease.

11. **Monotone Convergence Theorem**.

> **Monotone Convergence Theorem**
> If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \ge n_0$, then $\{a_n\}$ converges.