

6.3 Taylor and Maclaurin Polynomials (p.497-512)

1. Review: Sequences and Series:

A Sequence is an enumerated list of numbers and it can be denoted by $a_1, a_2, a_3, \dots$, or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Let $\{a_k\}$ be a sequence. The Series is the **sum** of all the term of a_k for $k \geq 1$: $\sum_{k=1}^{\infty} a_k$

Example: $\{1, 2, 100, 3, 4, 1000, \dots\}$ or $\{1, 4, 9, 16, 25, 36, 49, 64, \dots\}$

2. Definition: Power Series.

A series of the form is called a power series centered at $x=0$ if the form looks like: $\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$

A series of the form is called a power series centered at $x=a$ if the form looks like:

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + C_3 (x-a)^3 + C_4 (x-a)^4 + \dots$$

3. A Power Series Representation of a Function.

If a function $f(x)$ is good enough (?), then this function $f(x)$ has

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + C_3 (x-a)^3 + C_4 (x-a)^4 + \dots$$

Why do we need this? Divide and Conquer: Knowing Polynomials = knowing the approximation of $f(x)$.

4. How to find out the coefficients $c_0, c_1, c_2, c_3, c_4, c_5, \dots$ in the power series representation?

For c_0 , since $f(x) =$

then when $x = a$, we have

which implies

For c_1 , since $\frac{d}{dx} f(x) =$

then when , we have

which implies

For c_2 , since $\frac{d^2}{dx^2} f(x) =$

then when , we have

which implies

For c_3 , since $\frac{d^3}{dx^3} f(x) =$

then when , we have

which implies

For c_4 , since $\frac{d^4}{dx^4} f(x) =$

then when , we have

which implies

For c_n , we have

Therefore,

$f(x) =$

5. *Definition:* Taylor Series.

If a function f has _____, then _____ is

The Taylor Series for f at _____ is known as _____:

6. Examples. Find the Maclaurin Series for each given function.

(a) $f(x) = \sin(x)$

(b) $f(x) = \cos(x)$

(b) $f(x) = e^x$

7. *Definition:* Taylor Polynomials. Given a function f which has derivatives of all orders at $x = a$.

Then the n th partial sum of the Taylor series for f at $x = a$ is known as _____ or _____
_____:

The Taylor Polynomial for f at $x = 0$ is known as _____ or _____
_____:

8. Example. Find the Taylor Polynomial of degree 3 for $f(x) = \sin x$ at $a = \frac{\pi}{2}$.

9. Taylor's Theorem with Remainder.

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a . Let p_n be the n th Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the n th remainder. Then for each x in the interval I , there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

If there exists a real number M such that $|f^{(n+1)}(x)| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

for all x in I .