

MAT1575, Classwork11, Fall2026

3.7 Improper Integrals (p.288-300)

1.(Review) **Arrow Notation:** Given a constant a and we have

$x \rightarrow a^+$:	x approaches a from the right (x is very closed to a but $x \neq a$ and $x > a$)
$x \rightarrow a^-$:	x approaches a from the left (x is very closed to a but $x \neq a$ and $x < a$)
$x \rightarrow \infty$:	x approaches infinity (x increases without bound)
$x \rightarrow -\infty$:	x approaches negative infinity (x decreases without bound)

2. Evaluate each of the following integrals:

(a) $\int_1^{\infty} \frac{1}{x} dx$ ln|x| |₁[∞]

(b) $\int_0^4 \frac{1}{\sqrt{4-x}} dx$ ← [0, 4]
when $x \rightarrow 4^-$, $\frac{1}{\sqrt{4-x}} \rightarrow \infty$

(c) $\int_{-1}^1 \frac{1}{x^3} dx$
 $x \rightarrow 0^+$, $\frac{1}{x^3} \rightarrow \infty$
 $x \rightarrow 0^-$, $\frac{1}{x^3} \rightarrow -\infty$

3. Improper integrals: we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called Improper Integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

4. Case1: Integrating over an Infinite Interval

Evaluate (a) $\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$
 $= \lim_{t \rightarrow \infty} [\ln|x|]_1^t$
 $= \lim_{t \rightarrow \infty} [\ln t - \ln 1]$
 $= \lim_{t \rightarrow \infty} \ln t \rightarrow \infty$ or it diverges

(b) $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} [-\frac{1}{x}]_1^t$
 $= \lim_{t \rightarrow \infty} [-\frac{1}{t} - (-\frac{1}{1})] = \lim_{t \rightarrow \infty} [-\frac{1}{t} + 1]$
 $= 1$

5. Definition for Case1:

(a) Let $f(x)$ be continuous on interval $[a, \infty)$: $\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$ provides this limit exists
 (b) Let $f(x)$ be continuous on interval $(-\infty, b]$: $\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$ provides this limit exists

In each items, if limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

(c) Let $f(x)$ be continuous on interval $(-\infty, \infty)$: $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$

Then this improper integral is said to converge if both $\int_{-\infty}^0 f(x) dx$ and $\int_0^{\infty} f(x) dx$ exist

6. Evaluate $\int_{-\infty}^{\infty} x e^x dx = \int_{-\infty}^0 x e^x dx + \int_0^{\infty} x e^x dx$ $u=x, dv=e^x dx, du=dx, v=e^x$
 Check $\int_{-\infty}^0 x e^x dx = \lim_{t \rightarrow -\infty} \int_t^0 x e^x dx = \lim_{t \rightarrow -\infty} [x e^x - \int e^x dx] = \lim_{t \rightarrow -\infty} [x e^x - e^x]$
 $= \lim_{t \rightarrow -\infty} [t e^t - e^t - 1] = \lim_{t \rightarrow -\infty} [e^t (t-1) - 1]$
replace "t" by "-t"
 $= \lim_{t \rightarrow -\infty} [e^{-t} (-t-1) - 1] = \lim_{t \rightarrow -\infty} [-\frac{1}{e^t} (t+1) - 1] = -1$

7. Case2: Integrating a Discontinuous Integrand

Evaluate (a) $\int_0^4 \frac{1}{\sqrt{4-x}} dx = \lim_{t \rightarrow 4^-} \int_0^t \frac{1}{\sqrt{4-x}} dx = \lim_{t \rightarrow 4^-} [-2\sqrt{4-x}]_0^t$
 $= \lim_{t \rightarrow 4^-} [-2\sqrt{4-t} - (-2\sqrt{4-0})]$
 $= \lim_{t \rightarrow 4^-} [-2\sqrt{4-t} + 2\sqrt{4}]$
goes to 0 as $t \rightarrow 4^-$
 $= 2 \cdot 0 + 4 = 4$

$\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$
 $= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^3} dx + \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^3} dx$

$\int_0^{\infty} x e^x dx = \lim_{t \rightarrow \infty} \int_0^t x e^x dx$
 $= \lim_{t \rightarrow \infty} [x e^x - e^x]_0^t = \lim_{t \rightarrow \infty} [t e^t - e^t - 0 + 1]$
 $= \lim_{t \rightarrow \infty} [e^t (t-1) + 1]$
so it diverges
 Thus, $\int_{-\infty}^{\infty} x e^x dx$ diverges

8. Definition for Case2:

(a) Let $f(x)$ be continuous on interval $[a, b)$: $\int_a^b f(x) dx = \lim_{x \rightarrow b^-} \int_a^x f(x) dx$ provided this limit exists

(b) Let $f(x)$ be continuous on interval $(a, b]$: $\int_a^b f(x) dx = \lim_{x \rightarrow a^+} \int_x^b f(x) dx$

In each items, if the limit exists, then the improper integral converges. If the limit does not exist, then the improper integral is said to diverges.

(c) Let $f(x)$ be continuous on interval $[a, b]$ except at a point $c \in (a, b)$: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
Then this improper integral is said to converge if both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ exist

9. A Comparison Theorem: Alternative Way to Evaluate Improper Integral.

Consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \leq f(x) \leq g(x)$ for $x \geq a$. Then we have

$$0 \leq \int_a^t f(x) dx \leq \int_a^t g(x) dx \text{ for } t \geq a$$

(a) If $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx \rightarrow \infty$, then $\int_a^\infty g(x) dx \rightarrow \infty$

(b) If $\int_a^\infty g(x) dx = \lim_{t \rightarrow \infty} \int_a^t g(x) dx = L$, where L is a real number, then $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx < L$ (it converges)

10. Applying the Comparison Theorem.

Use a comparison to show that $\int_1^\infty \frac{1}{xe^x} dx$ converges

Since $0 \leq \frac{1}{xe^x} \leq \frac{1}{e^x} \quad x \in [1, \infty) \Rightarrow \int_1^\infty \frac{dx}{xe^x} \leq \int_1^\infty \frac{dx}{e^x}$

check $\int_1^\infty \frac{dx}{e^x} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{e^x} = \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx$

$= \lim_{t \rightarrow \infty} [-e^{-x}]_1^t = \lim_{t \rightarrow \infty} [-e^{-t} - (-e^{-1})] = \lim_{t \rightarrow \infty} [-\frac{1}{e^t} + \frac{1}{e}]$

$= \frac{1}{e}$ it converges $\Rightarrow \int_1^\infty \frac{dx}{xe^x}$ converges

goes to 0 as $t \rightarrow \infty$

11. Check if each given integral is an improper integral:

(a) $\int_0^\infty e^{-x} \cos x dx$ Improper

(b) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$ $x \rightarrow 0^+ \frac{\ln x}{\sqrt{x}} \rightarrow -\infty$ Improper

(c) $\int_1^5 \frac{1}{\sqrt{x-1}} dx$ $x \rightarrow 1^+, \frac{1}{\sqrt{x-1}} \rightarrow \infty$ Improper

(d) $\int_{-1}^2 \frac{dx}{x^3}$ $x \rightarrow 0^-$ or $x \rightarrow 0^+$ $\frac{1}{x^3} \rightarrow -\infty$ or $\frac{1}{x^3} \rightarrow \infty$ Improper

(e) $\int_3^5 \frac{5}{(x-4)^2} dx$ $x \rightarrow 4^-$ $\frac{5}{(x-4)^2} \rightarrow \infty$ $x \rightarrow 4^+$ $\frac{5}{(x-4)^2} \rightarrow \infty$ Improper