

3.4 Partial Fractions (p.262-270)

1. Evaluate each of the following integrals:

(a) $\int \left(\frac{1}{x+1} + \frac{2}{x-2} \right) dx$

$$= \int \frac{dx}{x+1} + \int \frac{2dx}{x-2}$$

$$= \ln|x+1| + 2 \ln|x-2| + C$$

(b) $\int \frac{3x}{x^2-x-2} dx = \int \frac{3x}{(x-2)(x+1)} dx$

Handwritten notes: $\frac{(x-2)}{(x-2)(x+1)} + \frac{2(x+1)}{(x-2)(x+1)} = \frac{3x}{(x+1)(x-2)}$

Handwritten notes: $3x^2 + 2x + 1$ deg 2 coeff 3

Handwritten notes: $5x^2 + 4x + 2x + 5$ deg 3 4

2. Rational Function. A rational function is a fraction $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials, and $g(x) \neq 0$ is not the zero function.

3. Degree of a Polynomial. The coefficient of the variable to the highest power in a polynomial $f(x)$, is called the leading coefficient and the highest power is called the degree of the polynomial, denoted by deg(f).

4. Example when $\deg(f(x)) \geq \deg(g(x))$: Evaluate $\int \frac{x^2+3x+5}{x+1} dx$

$$\int \frac{x^2+3x+5}{x+1} dx = \int \left(x+2 + \frac{3}{x+1} \right) dx$$

$$= \frac{x^2}{2} + 2x + 3 \ln|x+1| + C$$

Handwritten notes: $\frac{x^2+3x+5}{(x+1)} = \frac{(x+1)(x+2)+3}{(x+1)} = \frac{(x+1)(x+2)}{(x+1)} + \frac{3}{(x+1)}$

Handwritten notes: $x+2$

Handwritten notes: $\frac{x^2+3x+5}{x+1} \rightarrow \text{deg } 2$

Handwritten notes: $\frac{3}{x+1} \rightarrow \text{deg } 1$

Handwritten notes: $\frac{x^2+3x+5}{x+1} = \frac{(x+1)(x+2)+3}{(x+1)} = \frac{(x+1)(x+2)}{(x+1)} + \frac{3}{(x+1)}$

5. Partial Fraction Decomposition: To evaluate $\int \frac{f(x)}{g(x)} dx$ with $\deg(f) < \deg(g)$, we can separate the fraction $\frac{f(x)}{g(x)}$ into several rational functions and then solve them one by one.

6. Partial Fraction with Nonrepeated Linear Factors: $\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$

Evaluate $\int \frac{3x+2}{x^3-x^2-2x} dx = \int \frac{3x+2}{x(x-2)(x+1)} dx$

$$= \int \frac{-1}{x} + \frac{\frac{4}{3}}{x-2} + \frac{-\frac{1}{3}}{x+1} dx$$

$$= -\ln|x| + \frac{4}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C$$

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$= \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

To find A, try $x=0$, $2 = A(0-2)(0+1) \Rightarrow 2 = -2A, A = -1$

To find B, try $x=2$, $3 \cdot 2 + 2 = 0 + B \cdot 2 \cdot (2+1) \Rightarrow B = \frac{4}{3}$

To find C, try $x=-1$, $3(-1)+2 = 0 + 0 + C \cdot (-1) \cdot (-1-2) \Rightarrow C = -\frac{1}{3}$

7. Partial Fraction with Repeated Linear Factors: $\frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx$.

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{(x-1)} + \frac{B}{(2x-1)} + \frac{D}{(2x-1)^2}$$

$$= \int \frac{-1}{(x-1)} + \frac{2}{(2x-1)} + \frac{3}{(2x-1)^2} dx$$

$$= -\ln|x-1| + \frac{2}{2} \ln|2x-1| - \frac{3}{2} \frac{1}{2x-1} + C$$

To find A, $x=1$, $1-2 = A \cdot (2 \cdot 1 - 1)^2 \Rightarrow A = -1$
 To find D, $x = \frac{1}{2}$, $\frac{1}{2} - 2 = 0 + 0 + D \cdot (\frac{1}{2} - 1)^2$
 $-\frac{3}{2} = -\frac{1}{2} D \Rightarrow D = 3$
 To find B, try $x=0$, $0-2 = (-1) \cdot (2 \cdot 0 - 1)^2 + B(0-1)(2 \cdot 0 - 1) + 3 \cdot (-1)$
 $-2 = -1 + B - 3 \Rightarrow B = 2$

8. Partial Fraction with Irreducible Quadratic Factors: $\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2} + \dots + \frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$

Evaluate $\int \frac{2x-3}{x^3+x} dx$.

$$\frac{2x-3}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \int \frac{-3}{x} + \frac{3x+2}{x^2+1} dx$$

$$= -3 \ln|x| + \int \frac{3x}{x^2+1} dx + \int \frac{2}{x^2+1} dx$$

$$= -3 \ln|x| + \frac{3}{2} \ln|x^2+1| + 2 \tan^{-1}x + C$$

Find A, let $x=0$, $2 \cdot 0 - 3 = A(0^2+1) \Rightarrow A = -3$
 $0x^2 + (2x-3) = -3(x^2+1) + Bx^2 + Cx = (B-3)x^2 + Cx - 3$
 $B-3=0, B=3, C=2$

9. Express the rational function as a sum or difference of two or several simpler rational expressions.

(a) $\frac{x-3}{x+2}$

(b) $\frac{x^2+3x+1}{x^2-4}$

(c) $\frac{x+1}{(x+3)(x-2)}$

(d) $\frac{x-2}{(x+1)(x^2+4)}$

(e) $\frac{2x+3}{(x^2+1)(x^2+4)}$

(a) $\frac{x+2-5}{x+2} = \frac{x+2}{x+2} + \frac{-5}{x+2}$
 $= 1 - \frac{5}{x+2}$

(c) $\frac{A}{x+3} + \frac{B}{x-2}$
 $(A = \frac{2}{5}, B = \frac{3}{5})$

(d) $\frac{A}{x+1} + \frac{Bx+C}{x^2+4}$
 $(A = -\frac{3}{5}, B = \frac{3}{5}, C = \frac{2}{5})$

(b) $\frac{x^2-4 + 3x+5}{x^2-4} = \frac{x^2-4}{x^2-4} + \frac{3x+5}{x^2-4} = 1 + \frac{3x+5}{x^2-4}$

$x^2-4 + g(x) = x^2+3x+1$
 $g(x) = (x^2+3x+1) - (x^2-4)$
 $= 3x+5$

(e) $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$
 $(A = \frac{2}{3}, B = 1, C = -\frac{2}{3}, D = -1)$