

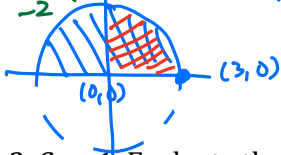
$$y = \sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x^2 + y^2 = 9$$

MAT1575, Classwork9, Fall2026

3.3 Trigonometric Substitution (p.250-259)

1. Evaluate each of the following integrals:

(a) $\int_{-2}^2 \sqrt{9-x^2} dx = \frac{9}{4}\pi$



(b) $\int \frac{x}{\sqrt{9-x^2}} dx$
 $u = 9-x^2, du = -2x dx; \frac{du}{-2} = x dx$
 $= \int \frac{1}{\sqrt{u}} \frac{du}{-2}$

(c) $\int x\sqrt{9-x^2} dx$
 $u = 9-x^2, du = -2x dx; \frac{du}{-2} = x dx$
 $= \int \sqrt{u} \cdot \frac{du}{-2}$

2. Case 1: Evaluate the given indefinite integral $\int \sqrt{9-x^2} dx$

Hint: let $x = 3 \sin \theta, dx = 3 \cos \theta d\theta$

$$\int \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta d\theta = \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta d\theta$$

$$= \int 3|\cos\theta| \cdot 3\cos\theta d\theta \stackrel{(*)}{=} \int 9\cos^2\theta d\theta = \int \frac{9}{2} + \frac{9\cos(2\theta)}{2} d\theta$$

$$= \frac{9}{2}\theta + \frac{9}{2} \frac{\sin(2\theta)}{2} + C = \frac{9}{2}\theta + \frac{9}{4} \sin\theta \cdot \cos\theta + C$$

$$= \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{4} \frac{x}{3} \frac{\sqrt{9-x^2}}{3} + C$$

*Check the angle based on x:

$$-3 \leq x \leq 3, -3 \leq 3\sin\theta \leq 3, -1 \leq \sin\theta \leq 1$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow \cos\theta \geq 0 (*)$$

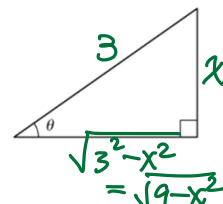
Under this range of θ , we have

*How to put x back to the result:

$$x = 3\sin\theta, \sin\theta = \frac{x}{3}$$

$$\theta = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}$$



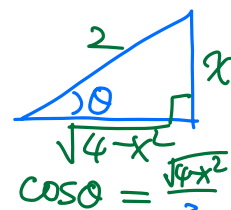
3. Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

(1) Make the substitution $x = a \cdot \sin\theta$ and $dx = a \cdot \cos\theta d\theta$ which yields $\sqrt{a^2-x^2} = a \cos\theta$

(2) Simplify the expression.

(3) Evaluate the integral using techniques from the section on trigonometric integrals.

(4) Use the reference triangle to rewrite the result in terms of x: $\theta = \sin^{-1}\left(\frac{x}{3}\right)$



4. Evaluate $\int \frac{\sqrt{4-x^2}}{x} dx$

$$x = 2\sin\theta, dx = 2\cos\theta d\theta \Rightarrow \sin\theta = \frac{x}{2}$$

$$= \int \frac{\sqrt{4-4\sin^2\theta}}{2\sin\theta} \cdot 2\cos\theta d\theta = \int \frac{2\cos\theta}{2\sin\theta} \cdot 2\cos\theta d\theta = \int \frac{2\cos^2\theta}{\sin\theta} d\theta$$

$$= \int \frac{2(1-\sin^2\theta)}{\sin\theta} d\theta = \int \frac{2}{\sin\theta} - 2\sin\theta d\theta = \int 2\csc\theta - 2\sin\theta d\theta$$

$$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C = 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + 2\frac{\sqrt{4-x^2}}{2} + C$$

5. Evaluate $\int x^3\sqrt{1-x^2} dx$ two ways: (1) using $u = 1-x^2$, (2) using a trigonometric substitution.

(1) $u = 1-x^2, du = -2x dx, x^2 = 1-u, \frac{du}{-2} = x dx$

$$\int x^3\sqrt{1-x^2} dx = \int x^2\sqrt{1-x^2} x dx$$

$$= \int (1-u) \cdot \sqrt{u} \frac{du}{-2}$$

$$= \int \frac{1}{-2} u^{\frac{1}{2}} - \frac{1}{2} u^{\frac{3}{2}} du = \int \frac{u^{\frac{1}{2}}}{-2} - \frac{u^{\frac{3}{2}}}{\frac{5}{2}} du$$

$$= \frac{2}{5} \cdot \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2}{3} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$\rightarrow \frac{(1-x^2)^{\frac{5}{2}}}{5} - \frac{(1-x^2)^{\frac{3}{2}}}{3} + C$$

(2) $x = \sin\theta, dx = \cos\theta d\theta$



$$\sin\theta = \frac{x}{1}, \cos\theta = \frac{\sqrt{1-x^2}}{1}$$

$$\int x^3\sqrt{1-x^2} dx$$

$$= \int \sin^3\theta \cdot \cos\theta \cdot \cos\theta d\theta$$

$$= \int \sin^3\theta \cdot \cos^2\theta d\theta$$

$$= \int \sin\theta \cdot \sin^2\theta \cdot \cos^2\theta d\theta$$

$$= \int (1-\cos^2\theta) \cdot \cos^2\theta \cdot \sin\theta d\theta$$

$$u\text{-sub} = -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5} + C$$

$$= -\frac{(\sqrt{1-x^2})^3}{3} + \frac{(\sqrt{1-x^2})^5}{5} + C$$

6. Case2: Evaluate the given indefinite integral $\int \frac{dx}{\sqrt{1+x^2}}$

$$= \int \frac{\sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|} = \int \frac{\sec^2 \theta}{\sec \theta} d\theta$$

Hint: let $x = \tan \theta$ $dx = \sec^2 \theta d\theta$
 *Check the angle based on x :
 $-\infty < x < \infty, -\infty < \tan \theta < \infty, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

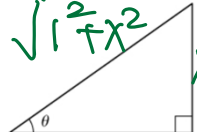
$$= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

Under this range of θ , we have

$$\sec \theta > 0$$

$$= \ln |\sqrt{1+x^2} + x| + C$$

*How to put x back to the result:
 $\tan \theta = \frac{x}{1}, \sec \theta = \frac{\sqrt{1+x^2}}{1}$



7. Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

(1) Make the substitution $x = a \cdot \tan \theta$ and $dx = a \cdot \sec^2 \theta d\theta$ which yields $\sqrt{a^2 + x^2} = a \cdot \sec \theta$

(2) Simplify the expression.

(3) Evaluate the integral using techniques from the section on trigonometric integrals.

(4) Use the reference triangle to rewrite the result in terms of x : $\theta = \tan^{-1}(x/a)$

9 $\sec^2 \theta - 9 = 9 \tan^2 \theta$
 Hint: let $x = 3 \sec \theta$ $dx = 3 \sec \theta \cdot \tan \theta d\theta$
 *Check the angle based on x :
 $x \in (-\infty, -3] \cup [3, \infty), -\infty < 3 \sec \theta \leq -3$ or $3 \sec \theta < \infty$

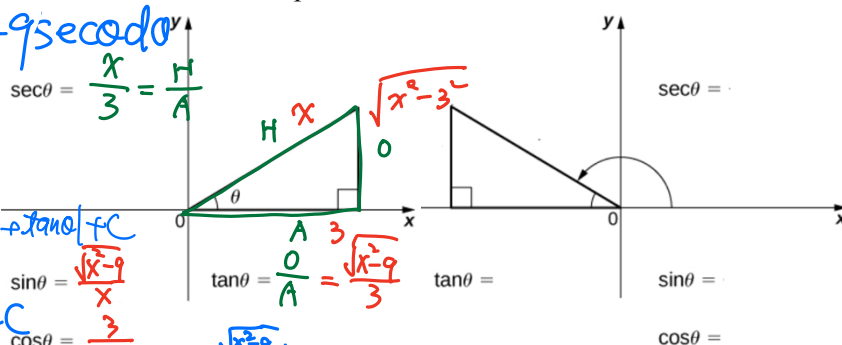
$-\infty < \sec \theta \leq -1$ or $1 \leq \sec \theta < \infty, -\frac{\pi}{2} < \theta \leq \pi, 0 \leq \theta < \frac{\pi}{2}$

Under this range of θ , we have

$$|\tan \theta| = \begin{cases} \tan \theta, & 0 \leq \theta < \frac{\pi}{2} \\ -\tan \theta, & \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

*How to put x back to the result:

$$\begin{aligned} & \int \sqrt{(3 \sec \theta)^2 - 9} \cdot 3 \sec \theta \cdot \tan \theta d\theta \\ &= \int 3 |\tan \theta| \cdot 3 \sec \theta \cdot \tan \theta d\theta \\ & \stackrel{0 \leq \theta < \frac{\pi}{2}}{=} \int 9 \tan^2 \theta \cdot \sec \theta d\theta \quad \stackrel{-\frac{\pi}{2} < \theta < \pi}{=} \int 9 \tan^2 \theta \cdot \sec \theta d\theta \\ &= \int 9 (\sec^2 \theta - 1) \sec \theta d\theta = \int 9 \sec^3 \theta - 9 \sec \theta d\theta \\ &= \int 9 \sec^3 \theta d\theta - \int 9 \sec \theta d\theta \\ &= \frac{9}{2} \sec \theta \tan \theta + \frac{9}{2} \ln |\sec \theta + \tan \theta| - 9 \ln |\sec \theta + \tan \theta| + C \\ &= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{x^2-9}}{3} - \frac{9}{2} \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C \end{aligned}$$



9. Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{x^2 - a^2}$

(1) Make the substitution $x = a \cdot \sec \theta$ and $dx = a \cdot \sec \theta \cdot \tan \theta d\theta$ which yields $\sqrt{x^2 - a^2} = |a \cdot \tan \theta|$

(2) Simplify the expression.

(3) Evaluate the integral using techniques from the section on trigonometric integrals.

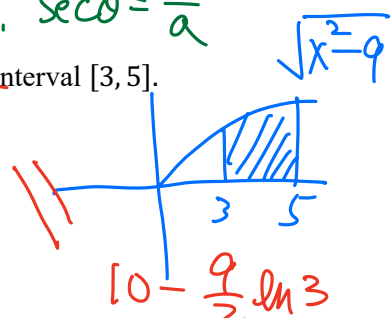
(4) Use the reference triangle to rewrite the result in terms of x : $\theta = \sec^{-1}(x/a)$, $\sec \theta = x/a$

10. Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and x -axis over the interval $[3, 5]$.

$$\int_3^5 \sqrt{x^2 - 9} dx = \left[\frac{9}{2} \frac{x}{3} \cdot \frac{\sqrt{x^2-9}}{3} - \frac{9}{2} \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| \right]_3^5$$

let $x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta$

$3 < 3 \sec \theta < 5, 1 < \sec \theta < \frac{5}{3}, 0 < \theta < \frac{\pi}{2}$



$10 - \frac{9}{2} \ln 3$