MAT1575, Classwork3, Fall2026

$\Delta x = \frac{x_n - x_0}{n}$

1.2 The Definite Integral (p.24-36)

a $\quad a+\Delta x \quad a+2\Delta x \quad \to b = a + n\cdot\Delta x$

1. *Review*: Let f be defined on a closed interval $[a,b]$ and $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a regular partition of $[a,b]$. Let Δx be the width of each subinterval. Then $\Delta x = \frac{b-a}{n}$, $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$, and $R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x$

2. Forming *Riemann Sums*: Following the assumption in 1., let x_i^* be any point in $[x_{i-1}, x_i]$.

A Riemann Sum is defined for $f(x)$ as $\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$

Then, the area under the curve $y = f(x)$ on $[a,b]$ is given by $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

3. *Upper Sum and Lower Sum*: Depending on the choices of x_i^* in $[x_{i-1}, x_i]$, we have, for $i = 1,2,\dots,n$,

(a) The Riemann Sum is called upper sum if $f(x_i^*) \geq f(x)$ which is the maximum value of $f(x)$ for $x \in [x_{i-1}, x_i]$.

(b) The Riemann Sum is called lower sum if $f(x_i^*) \leq f(x)$ which is the minimum value of $f(x)$ for $x \in [x_{i-1}, x_i]$.

Then, we have lower sum $\leq$ Area of A $\leq$ Upper Sum.

4. *Definition*: Definite Integral.

If $f(x)$ is a function defined on an interval $[a,b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If this limit exists, $f(x)$ is said to be integrable on $[a,b]$, or is an integrable function.
The numbers a and b are x-values and are called limits of integration; a is the lower limit and b is upper limit. Function $f(x)$ is called the integrand, and the dx indicates that is a function with respect to x, called the variable of integration.

5. What is the difference between definite integral and indefinite integral?

A definite integral is a number. An indefinite integral is a family of functions

6. *Example*: Evaluating an Integral Using the Definition.

(a) Use the definition of the definite integral to evaluate $\int_0^8 (x+2)dx$ $=48$. Use a right endpoint approximation to generate the Rieman sum.

$f(x) = x+2$, $I = [0,8]$, $\Delta x = \frac{8}{n}$, $P = \{0,\ 0+\Delta x,\ 0+2\Delta x,\ 0+3\Delta x, \dots, 0+(n-1)\Delta x,\ 8\}$ $(=0+n\cdot\Delta x)$ and $x_i = 0 + i\cdot\Delta x = \frac{8}{n}i$, $i = 0,1,2,\dots,n$

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{8}{n}\cdot i\right)\frac{8}{n} = \sum_{i=1}^{n}\left(\frac{8}{n}\cdot i + 2\right)\frac{8}{n} = \sum_{i=1}^{n}\left(\frac{64}{n^2}\cdot i + \frac{16}{n}\right) = \sum_{i=1}^{n}\frac{64}{n^2}i + \sum_{i=1}^{n}\frac{16}{n}\cdot 1 = \frac{64}{n^2}\sum_{i=1}^{n} i + \frac{16}{n}\sum_{i=1}^{n} 1 = \frac{64}{n^2}\cdot\frac{n(n+1)}{2} + 16$

$\underbrace{\frac{16}{n} + \frac{16}{n} + \frac{16}{n} + \cdots + \frac{16}{n}}_{n\text{-times}}$

$= \frac{32}{n^2}(n^2+n) + 16 = \frac{32}{n^2}n^2 + \frac{32}{n^2}\cdot n + 16 = 32 + \frac{32}{n} + 16$

$\int_0^8 (x+2)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(48 + \frac{32}{n}\right) = 48$

(b) Use the definition of the definite integral to evaluate $\int_{-1}^{3} (2x-1)dx$. Use a right endpoint approximation to generate the Rieman sum.

$f(x) = 2x-1$, $I = [-1,3]$, $\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$, $P = \{-1,\ -1+\Delta x,\ -1+2\Delta x,\ -1+3\Delta x, \dots, -1+(n-1)\Delta x,\ -1+n\Delta x = 3\}$ and $x_i = -1 + i\frac{4}{n}$, $i = 0,1,2,\dots,n$

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[2\left(-1+\frac{4}{n}i\right) - 1\right]\cdot\frac{4}{n} = \sum_{i=1}^{n} \frac{8}{n}\left(-1+\frac{4}{n}i\right) - \frac{4}{n} = \sum_{i=1}^{n} -\frac{8}{n} + \frac{32}{n^2}i - \frac{4}{n}$

$= \sum_{i=1}^{n}\left(\frac{32}{n^2}i - \frac{12}{n}\right)$

$= \sum_{i=1}^{n}\frac{32}{n^2}i - \sum_{i=1}^{n}\frac{12}{n}$

$= \frac{32}{n^2}\sum_{i=1}^{n} i - 12$

$16 + \frac{16}{n} - 12$

$= 4 + \frac{16}{n}$

$\int_{-1}^{3}(2x-1)dx = \lim_{n\to\infty}\left(4 + \frac{16}{n}\right) = 4$

$(x-a)^2+(y-b)^2=r^2 \Rightarrow$ center (a,b), radius $=r$

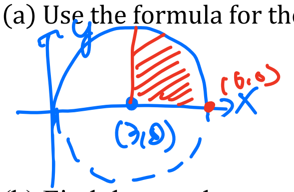
7. *Example*: Using Geometric Formulas to Calculate Definite Integrals.

$y=\sqrt{9-(x-3)^2}$ $(y\geq 0)$

$y^2=9-(x-3)^2 \Rightarrow (x-3)^2+y^2=9$

center $(3,0)$, radius $=3$

(a) Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.



$\int_3^6 \sqrt{9-(x-3)^2}\,dx$

$=\frac{\pi\cdot 3^2}{4}=\frac{9}{4}\pi$

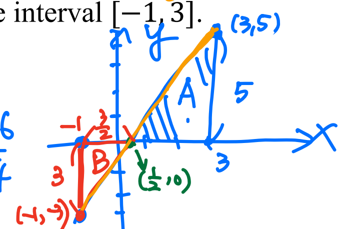
$y=2x-1$

(b) Find the area between the curve of the function $f(x)=2x-1$ and the x-axis over the interval $[-1,3]$.

$\int_{-1}^{3}(2x-1)\,dx = A - B$

$=\frac{1}{2}\left(5\cdot\frac{5}{2}\right)-\frac{1}{2}\left(3\cdot\frac{3}{2}\right)=\frac{25}{4}-\frac{9}{4}=\frac{16}{4}$

$=4$



9. *Definition*: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a,b]$. Let A_1 be the area between $f(x)$ above the x-axis and the x-axis, and let A_2 be the area between $f(x)$ below the x-axis and the x-axis. Then

(a) the net signed area between $f(x)$ and x axis is given by $\int_a^b f(x)dx =$ $A_1 - A_2$

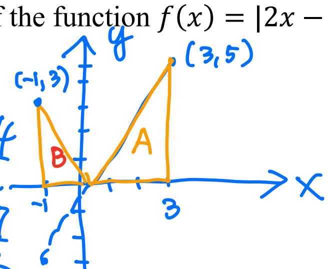
(b) the total area between $f(x)$ and x axis is given by $\int_a^b |f(x)|dx =$ $A_1 + A_2$

10. *Example*. Find the area between the curve of the function $f(x)=|2x-1|$ and the x-axis over the interval $[0,3]$. -1

$y=|2x-1|$ on $[-1,3]$

$\int_{-1}^{3}|2x-1|\,dx = A+B=\frac{25}{4}+\frac{9}{4}=\frac{34}{4}$

$=\frac{17}{2}$



11. Properties of the Definite Integral.

(a) If the limits of integration are the same, the integral is just a line and contains no area: $\int_a^a f(x)dx =$ 0

$b\geq a$

(b) If the limits are reversed, then place a negative sign in front of the integral: $\int_b^a f(x)dx =$ $-\int_a^b f(x)dx$

(c) The integral of a sum is the sum of the integrals: $\int_a^b [f(x)+g(x)]dx =$ $\int_a^b f(x)dx+\int_a^b g(x)dx$

(d) The integral of a difference is the difference of the integrals: $\int_a^b [f(x)-g(x)]dx =$ $\int_a^b f(x)dx-\int_a^b g(x)dx$

(e) The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

$\cancel{\int_a^b [f(x)+g(x)]dx =}$ $\int_a^b k\cdot f(x)dx = k\int_a^b f(x)dx$

(f) $\int_a^b f(x)dx =$ $\int_a^c f(x)dx+\int_c^b f(x)dx$

12. Use the properties of the definite integral to express the definite integral of $f(x)=6x^3-4x^2+2x-3$ over the interval $[1,3]$ as the sum of four definite integrals.

$\int_1^3 (6x^3-4x^2+2x-3)dx = \int_1^3 6x^3dx - \int_1^3 4x^2dx + \int_1^3 2x\,dx - \int_1^3 3\,dx$

13. If it is known that $\int_1^5 f(x)dx = -3$ and $\int_2^5 f(x)dx = 4$, find the value of $\int_1^2 f(x)dx$.

$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^5 f(x)dx \Rightarrow \int_1^2 f(x)dx = \int_1^5 f(x)dx - \int_2^5 f(x)dx$

$=-3-4=-7$

$[0,1]$ $\quad \int_0^1 f(x)dx = \int_0^{\frac{1}{2}} f(x)dx + \int_{\frac{1}{2}}^1 f(x)dx \rightarrow c=\frac{1}{2}$

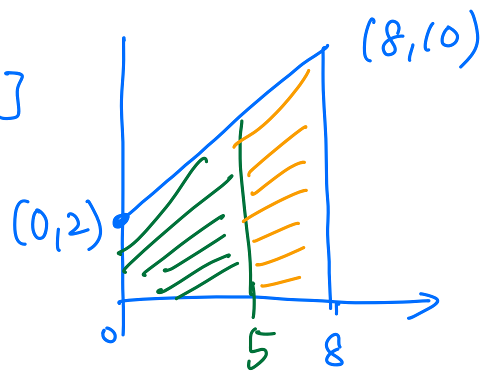
$= \int_0^{\frac{1}{3}} f(x)dx + \int_{\frac{1}{3}}^1 f(x)dx \rightarrow c=\frac{1}{3}$

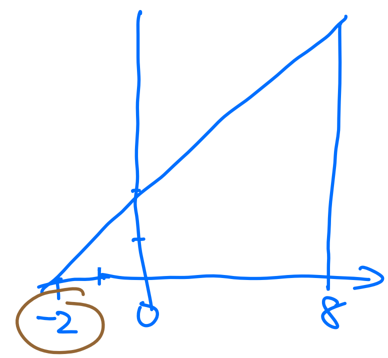
$= \int_0^{-1} f(x)dx + \int_{-1}^1 f(x)dx \quad c=-1$

$= -\int_{-1}^0 f(x)dx + \int_{-1}^1 f(x)dx$

$= \int_{-1}^1 f(x)dx - \int_{-1}^0 f(x)dx$

$f(x)=x+2 \quad [0,8]$





$\int_0^8 (x+2)dx = 48$

$\int_0^5 (x+2)dx + \int_5^8 (x+2)dx$

$c=5$

$\int_0^8 (x+2)dx$

$= \int_0^{-2} (x+2)dx + \int_{-2}^8 (x+2)dx$

$= -\int_{-2}^0 (x+2)dx + \int_{-2}^8 (x+2)dx$