Section 1.7

1. Use a direct proof to show that the sum of two odd integers is even.

Proof: Let a.b be two odd integers. We may assume that

$a = 2n+1$ and $b = 2m+1$ where m, n are integers.

Then $a+b = (2n+1)+(2m+1)$

$= 2n+2m+2 = 2(n+m+1)$

since a+b is a product of two and an integer $n+m+1$, then a+b is an even number.

2. Use a direct proof to show that the sum of two even integers is even.

Proof: Let a.b be two even integers. We may assume that

$a = 2n$ and $b = 2m$ where m, n are integers.

Then $a+b = 2n+2m$

$= 2(n+m)$

since a+b is a product of two and an integer $n+m$, then a+b is an even number.

3. Show that the square of an even number is an even number using a direct proof.

Proof: Let a be an even number, we may assume that

$a = 2n$ where n is an integer.

Then $a^2 = (2n)^2 = 4n^2 = 2(2n^2)$

since a^2 is a product of two and an integer $2n^2$, then a^2 is an even number.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

Proof: Let a be an even number, we may assume that $a=2n$ where n is an integer.
Then the negative of a, denote $-a$, is $-2n$
since $-a$ is a product of two and an integer $(-n)$
then $-a$ is an even number.

9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Proof: To prove this theorem by contradiction, we assume that the sum of an irrational number and a rational number is rational

Let $r=\frac{a}{b}$ be a rational number with $b\neq 0$, and s is an irrational number.

We have $r+s=\frac{c}{d}$ which is also a rational number with $d\neq 0$.

Since $r+s=\frac{c}{d}$, then we have

$$s=\frac{c}{d}-r \underset{r=\frac{a}{b}}{\uparrow =} \frac{c}{d}-\frac{a}{b}=\frac{cb-ad}{db}$$

which is also a rational number with $db\neq 0$. $\Rightarrow$ s is rational.

However, s is given as an irrational number and there is a contradiction since s cannot be both rational and irrational.

Therefore, the assumption is false and the sum of an irrational number and a rational number is irrational.

10. Use a direct proof to show that the product of two rational numbers is rational.

Proof: Let r, s be two rational numbers. we have

$r = \frac{a}{b}$ and $s = \frac{c}{d}$ with $b \neq 0, d \neq 0$.

Then $r \cdot s = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ is rational with $bd \neq 0$.

Therefore, the product of two rational numbers is rational.

11. Prove or disprove that the product of two irrational numbers is irrational.

We can disprove this by providing a counterexample:

let $a = \sqrt{2}$, $b = \sqrt{8}$ be two irrational numbers.

Then we have $a \cdot b = \sqrt{2} \cdot \sqrt{8} = \sqrt{16} = 4$ which is a rational number.

and it means that "the product of two irrational numbers is irrational"

is not true.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Proof: To prove this theorem by contradiction, we assume that

the product of an irrational number and a nonzero rational number is rational

Let $r = \frac{a}{b}$ be a rational number with $a \neq 0$, $b \neq 0$, and s is an irrational number.

We have $r \cdot s = \frac{c}{d}$ which is also a rational number with $d \neq 0$.

Since $r \cdot s = \frac{c}{d}$, then we have

$$s = \frac{c}{d} \cdot \frac{1}{r} = \frac{c}{d} \cdot \frac{b}{a} = \frac{cb}{da}$$

which is also a rational number with $da \neq 0$. $\Rightarrow s$ is rational.

However, s is given as an irrational number and there is a contradiction since S cannot be both rational and irrational.

Therefore, the assumption is false and the product of an irrational number and a nonzero rational number is irrational.

17. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Proof:

The contraposition of " if $\overset{p}{\boxed{x+y\geq 2}}$, then $\overset{q}{\boxed{x\geq 1 \text{ or } y\geq 1}}$" is " if $\underbrace{x<1 \text{ and } y<1}_{\neg q}$, then $\underbrace{x+y<2}_{\neg p}$" where x and y are real.

If $x<1$ and $y<1$, we have

$$x<1 \Rightarrow x+y<1+y \Rightarrow x+y<1+y \underset{\text{since } y<1}{<} 1+1=2$$

Therefore, $x<1$ and $y<1$ imply $x+y<2$.

18. Prove that if m and n are integers and mn is even, then m is even or n is even.

Proof: The contraposition of " if $\overset{p}{\boxed{mn \text{ is even}}}$, then $\overset{q}{\boxed{m \text{ is even or } n \text{ is even}}}$" is " if $\underbrace{m \text{ is odd and } n \text{ is odd}}_{\neg q}$, then $\underbrace{mn \text{ is odd}}_{\neg p}$" where m, n are integers

If m is odd and n is odd, we have

$m=2k+1$ and $n=2l+1$ where k, l are integers.

Then $m \cdot n = (2k+1)(2l+1) = 2k \cdot 2l + 2k + 2l + 1$

$= 2(kl + k + l) + 1$

and it is a product of two and an integer $kl+k+l$ adding 1 which is an odd number.

Therefore, if both m, n are odd, then mn is odd.