Section 5.1

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

Sol: Let $P(k)$ be the train stops at the k^{th} station.

Since the train stops at the first station ($P(1)$ is true) and the train stops at a station and it stops at the next as well ($P(k) \rightarrow P(k+1)$)

Then, by the rule of influence, we have

$$\begin{array}{l} P(1) \rightarrow P(2) \\ P(1) \\ \hline \therefore P(2) \end{array} \qquad \begin{array}{l} P(2) \rightarrow P(3) \\ P(2) \\ \hline \therefore P(3) \end{array} \qquad \cdots\cdots$$

which means that the train will stops at all the stations.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

$P(1)$ is " $1^2 = \frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6}$ "

b) Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n.

The left hand side of $P(1)$ is $1^2 = 1$ and the right hand side of $P(1)$ is $\frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6} = \frac{1\cdot2\cdot3}{6} = 1$

$\Rightarrow$ so $P(1)$ is true.

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

c) $P(n)$ is " $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ " and we assume it is true

d) We need to prove $P(n)$ implies $P(n+1)$

e) The left hand side of $P(n+1)$ is

$$\underbrace{1^2+2^2+3^2+\cdots+n^2}_{\frac{n(n+1)(2n+1)}{6}}+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

by $\underline{\underline{c)}}$,

$$= (n+1)\left[\frac{n(2n+1)}{6} + n+1\right]$$

$$= (n+1)\left[\frac{n(2n+1)}{6} + \frac{6(n+1)}{6}\right] = (n+1)\left[\frac{2n^2+n+6n+6}{6}\right]$$

$$= (n+1)\left[\frac{2n^2+7n+6}{6}\right] = (n+1)\,\frac{(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

f) By the rule of influence, since $P(1)$ is true and $P(n) \to P(n+1)$ for all positive integers n, then $P(n)$ is true for all positive integers n.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

Proof: Let $P(n)$ be " $1^2+3^2+5^2+\cdots+(2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ ".

① $P(0)$ is " $1^2 = \frac{(0+1)(2\cdot 0+1)(2\cdot 0+3)}{3}$ " and we have

$\frac{(0+1)(2\cdot 0+1)(2\cdot 0+3)}{3} = \frac{1\cdot 1\cdot 3}{3} = 1$ and it implies $P(0)$ is true.

② Assume $P(k)$ is true:
" $1^2+3^2+\cdots+(2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ " is true.

③ To show $P(k) \to P(k+1)$, we have
the left hand side of $P(k+1)$ is

$$\underbrace{1^2+3^2+\cdots+(2k+1)^2}_{\text{By ②, } \frac{(k+1)(2k+1)(2k+3)}{3}} + (2k+3)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

$$= (2k+3)\left[\frac{(k+1)(2k+1)}{3} + (2k+3)\right]$$

$$= (2k+3)\left[\frac{2k^2+3k+1}{3} + \frac{3(2k+3)}{3}\right] = (2k+3)\left(\frac{2k^2+3k+1+6k+9}{3}\right)$$

$$= (2k+3)\left(\frac{2k^2+9k+10}{3}\right) = \frac{(2k+3)\cdot(k+2)(2k+5)}{3}$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3} = \frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}$$

$\Rightarrow$ By induction, based on ①,②,③ $P(n)$ is true for all $n \in \mathbb{N}$.

7. Prove that $3 + 3\cdot 5 + 3\cdot 5^2 + \cdots + 3\cdot 5^n = 3(5^{n+1}-1)/4$ whenever n is a nonnegative integer.

Proof: Let $P(n)$ be " $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = \frac{3(5^{n+1}-1)}{4}$ "

① $P(0)$ is " $3 = \frac{3(5^{0+1}-1)}{4}$ " and we have

$\frac{3\cdot(5^{0+1}-1)}{4} = \frac{3\cdot(5^1-1)}{4} = \frac{3\cdot 4}{4} = 3$ and $P(0)$ is true.

② Assume $P(k)$ is true:

"$3+3\cdot5+\cdots+3\cdot5^k = \dfrac{3(5^{k+1}-1)}{4}$" is true

③ To show $P(k)\to P(k+1)$, we have

The left hand side of $P(k+1)$ =

$$\underbrace{3+3\cdot5+\cdots+3\cdot5^k}_{\frac{3(5^{k-1}-1)}{4}}+3\cdot5^{k+1} = \frac{3\cdot(5^{k+1}-1)}{4}+3\cdot5^{k+1}$$

$$= \frac{3\cdot(5^{k-1}-1)+4\cdot3\cdot5^{k+1}}{4}$$

$$= \frac{3(1\cdot5^{k+1}+4\cdot5^{k+1}-1)}{4} = \frac{3(5\cdot5^{k+1}-1)}{4} = \frac{3(5^{k+2}-1)}{4}$$

$\Rightarrow$ By induction, based on ①,②,③ $P(n)$ is true for all $n\in\mathbb{N}$.

9. a) Find a formula for the sum of the first n even positive integers.

b) Prove the formula that you conjectured in part (a).

a) $2+4+6+\cdots+2n = 2(1+2+\cdots+n) = 2\cdot\dfrac{n\cdot(n+1)}{2} = n(n+1)$.

b) Prove "$2+4+6+\cdots+2n = n(n+1)$" is true for all $n\in\mathbb{Z}^+$

<u>Proof</u>: Let $P(n)$ be "$2+4+6+\cdots+2n = n(n+1)$"

① $P(1)$ is "$2 = 1\cdot(1+1)$" and we have $1\cdot(1+1) = 1\cdot2 = 2$.

Thus $P(1)$ is true.

② Assume $P(k)$ is true:

"$2+4+6+\cdots+2k = k(k+1)$" is true

③ To show $P(k)\to P(k+1)$, we have

the left hand side of $P(k+1)$ is

$$\underbrace{2+4+6+\cdots+2k}_{k(k+1)}+2k+2 = k(k+1)+2k+2$$

$$= k^2+k+2k+2$$

$$= k^2+3k+2$$

$= (k+1)(k+2) = (k+1)(k+1+1)$

$\Rightarrow$ By induction, based on ①,②,③ $P(n)$ is true for all $n \in \mathbb{Z}^+$

11. a) Find a formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

Check Exam3 Question 5.

15. Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

Let $P(n)$ be " $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ "

① To show $P(1)$ is true, we have $P(1)$: " $1 \cdot 2 = \frac{1 \cdot (1+1) \cdot (1+2)}{3}$ "

and the left hand side $P(1)$ is $1 \cdot 2 = 2$ and

the right hand side $P(1)$ is $\frac{1 \cdot (1+1) \cdot (1+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$

$\Rightarrow P(1)$ is true

② Assume $P(k)$ is true:

" $1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$ " is true.

③ To show $P(k) \to P(k+1)$, we have

the left hand side of $P(k+1)$ is

$$\underbrace{1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1)}_{\frac{k(k+1)(k+2)}{3}} + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right]$$

$$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$$

$$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$$

$\Rightarrow$ By induction, based on ①,②,③ $P(n)$ is true for all $n \in \mathbb{Z}^+$

21. Prove that $2^n > n^2$ if n is an integer greater than 4.

Proof: Let $P(n)$ be "$n^2 < 2^n$" $n > 4$ (so $n = 5$ to begin with)

① To show $P(5)$ is true: $P(5)$ is "$2^5 > 5^2$"

The right hand side of $P(5) = 2^5 = 32$

The left hand side of $P(5) = 5^2 = 25$

$\Rightarrow$ $32 > 25$ and $P(5)$ is true.

② Assume $P(k)$ is true:

"$k^2 < 2^k$" is true for $k > 4$.

③ To show $P(k) \Rightarrow P(k+1)$, we have

the left hand side of $P(k+1)$ is

$$\boxed{\begin{array}{l} 2k+1 < 2^k \\ \text{for } k > 3 \end{array}}$$

$$\begin{aligned}(k+1)^2 = \underbrace{k^2}_{< 2^k} + 2k + 1 &< 2^k + \underline{2k+1} \\ &< 2^k + \underline{2^k} \\ &= 2 \cdot 2^k = 2^{k+1}\end{aligned}$$

$\Rightarrow$ By induction, based on ①,②,③ $P(n)$ is true for all integers $n > 4$.