Section 3.2

In Exercises 1–14, to establish a big-O relationship, find witnesses C and k such that $|f(x)| \le C|g(x)|$ whenever $x > k$.

1. Determine whether each of these functions is $O(x)$.

a) $f(x) = 10$ **b)** $f(x) = 3x + 7$
c) $f(x) = x^2 + x + 1$ **d)** $f(x) = 5 \log x$
e) $f(x) = \lfloor x \rfloor$ **f)** $f(x) = \lceil x/2 \rceil$

Sol: a) $|10| < 1 \cdot |x|$ for $x > 10$ $\Rightarrow$ $f(x)$ is $O(x)$ with $C = 1$, $k = 10$

b) $|3x+7| < |3x + x| < 4|x|$ for $x > 7$ $\Rightarrow$ $f(x)$ is $O(x)$ with $C = 4$, $k = 7$

c) $|f(x)| = |x^2+x+1| < |x^2+x^2+x^2| < 3|x^2|$ for $x > 1$ $\Rightarrow$ f is $O(x^2)$, not $O(x)$.

d) $|f(x)| = |5\log(x)| < |5x| < 5|x|$ for $x > 1$ $\Rightarrow$ $f(x)$ is $O(x)$ with $C = 5$, $k = 1$

e) $|f(x)| = |\lfloor x \rfloor| < 1 \cdot |x|$ for $x > 0$ $\Rightarrow$ $f(x)$ is $O(x)$ with $C = 1$, $k = 0$

f) $|f(x)| = \left|\left\lceil \frac{x}{2} \right\rceil\right| < 1 \cdot |x|$ for $x > 2$ $\Rightarrow$ $f(x)$ is $O(x)$ with $C = 1$, $k = 2$

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$.

Sol: $|f(x)| = |x^4 + 9x^3 + 4x + 7| < |x^4 + 9x^4 + 4x^4 + 7x^4| < |21x^4| < 21|x^4|$, $x > 1$

$\Rightarrow$ $f(x)$ is $O(x^4)$ with $C = 21$, $k = 1$.

5. Show that $(x^2 + 1)/(x + 1)$ is $O(x)$.

Sol: $\left|\frac{x^2+1}{x+1}\right| < \left|\frac{x^2+1}{x}\right| < \left|\frac{x^2+x^2}{x}\right| < \left|\frac{2x^2}{x}\right| < 2|x|$, for $x > 1$

$\Rightarrow$ $\frac{x^2+1}{x+1}$ is $O(x)$ with $C = 2$, $k = 1$.

7. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a) $f(x) = 2x^3 + x^2 \log x$
b) $f(x) = 3x^3 + (\log x)^4$
c) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
d) $f(x) = (x^4 + 5 \log x)/(x^4 + 1)$

Sol:

a) $|f(x)| < |2x^3| + |x^2 \log(x)| \underset{\log(x)<x}{\leq} |2x^3| + |x^2 \cdot x| < 3|x^3|$, $x > 1$

$\Rightarrow f(x)$ is $O(x^3)$ with $C=3$, $k=1$

b) $|f(x)| < |3x^3| + |(\log(x))^4| \underset{(\log(x))^4 < x^3}{<} |3x^3| + |x^3| < 4|x^3|$, $x > 1$

$\Rightarrow f(x)$ is $O(x^3)$ with $C=4$. $k=1$

c) $|f(x)| = \left|\frac{x^4+x^2+1}{x^3+1}\right| < \left|\frac{x^4+x^4+x^4}{x^3}\right| < 3\left|\frac{x^4}{x^3}\right| = 3|x|$, $x>1$

$x^2<x^4$; $1<x^4$; thus the numerator gets larger
$x^3+1 > x^3$; thus the denominator gets smaller

$\Rightarrow f(x)$ is $O(x^1)$ with $C=3$, $k=1$

d) $|f(x)| = \left|\frac{x^4+5\log(x)}{x^4+1}\right| \underset{\log(x)<x^4}{<} \left|\frac{x^4+5x^4}{x^4}\right| < 6\left|\frac{x^4}{x^4}\right| = 6$, $x>1$

$\Rightarrow f(x)$ is $O(1)$ with $C=6$, $k=1$

9. Show that $x^2 + 4x + 17$ is $O(x^3)$ but that x^3 is not $O(x^2 + 4x + 17)$.

Sol: Show that $x^2+4x+17$ is $O(x^3)$:

$|x^2+4x+17| < |x^2+4x^2+17x^2| < 22|x^2| < 22|x^3|$, $x>1$

$\Rightarrow$ $x^2+4x+17$ is $O(x^3)$ with $C=22$, $k=1$

show x^3 is NOT $O(x^2+4x+17)$:

$$\lim_{x\to\infty} \frac{x^3}{x^2+4x+17} \underset{L'(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{3x^2}{2x+4} \underset{L'(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{6x}{2} \to \infty$$ and

it means that we cannot find a C and k such that

$|x^3| < C|x^2+4x+17|$ for $x>k$ $\Rightarrow$ x^3 is NOT $O(x^2+4x+17)$.

11. Show that $3x^4 + 1$ is $O(x^4/2)$ and $x^4/2$ is $O(3x^4 + 1)$.

Sol: Show that $3x^4+1$ is $O(\frac{x^4}{2})$:

$|3x^4+1| \underset{1<x^4}{<} |3x^4+x^4| < 4|x^4| < 8\left|\frac{x^4}{2}\right|$, $x>1$.

$\Rightarrow$ $3x^4+1$ is $O(\frac{x^4}{2})$ with $C=8$, $k=1$.

Show that $\frac{x^4}{2}$ is $O(3x^4+1)$:

$\left|\frac{x^4}{2}\right| < |x^4| < |3x^4| < |3x^4+1|$, $x>1$

$\Rightarrow$ $\frac{x^4}{2}$ is $O(3x^4+1)$ with $C=1$, $k=1$.

13. Show that 2^n is $O(3^n)$ but that 3^n is not $O(2^n)$. (Note that this is a special case of Exercise 60.)

Sol: show that 2^n is $O(3^n)$:

$|2^n| < |3^n|$, $n>1$ $\Rightarrow$ 2^n is $O(3^n)$ with $C=1$, $k=1$

To show 3^n is not $O(2^n)$, we assume 3^n is $O(2^n)$, it means that there is a $C>0$ such that

$3^n \leq c \cdot 2^n$ for a $n > k$.

It implies that $\left(\frac{3}{2}\right)^n \leq c$ for a $n > k$.

However, since $\frac{3}{2} > 1$, there is no upper bound for $\left(\frac{3}{2}\right)^n$ and such c won't exist, a contradicition.

$\Rightarrow$ 3^n is not $O(2^n)$

15. Explain what it means for a function to be $O(1)$.

If a function $f(n)$ is $O(1)$, it means that there is a (C, k) s.t.

$$|f(n)| \leq C \cdot 1 \quad \text{for } n > k.$$

It implies that $f(n)$ is bounded for sufficiently large n.

17. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$. Show that $f(x)$ is $O(h(x))$.

Sol: If $f(x)$ is $O(g(x))$, then there is a (C_1, k_1) such that

$$|f(x)| \leq C_1 \, |g(x)| \text{ for } x > k_1 \quad —①$$

If $g(x)$ is $O(h(x))$, then there is a (C_2, k_2) $k_2 > k_1$

such that $|g(x)| \leq C_2 \, |h(x)|$ for $x > k_2$ —②.

Then, by ①, ②, we have C_1, C_2, and k_2 such that

$$|f(x)| \leq C_1 \, |g(x)| \leq C_1 \cdot C_2 \, |h(x)|, \quad x > k_2.$$

$$\Rightarrow |f(x)| \leq C_1 C_2 \, |h(x)|, \quad x > k_2$$

Therefore $f(x)$ is $O(h(x))$ with $C_1 C_2$ and $k = k_2$.

19. Determine whether each of the functions 2^{n+1} and 2^{2n} is $O(2^n)$.

For 2^{n+1}, we have $|2^{n+1}| \leq |2 \cdot 2^n| \leq 2|2^n|$, $n > 1$

$\Rightarrow$ 2^{n+1} is $O(2^n)$ with $C = 2$, $k = 1$

For 2^{2n}, we have $2^{2n} = (2^2)^n = 4^n$. since $4 > 2$, then

$4^n > 2^n$ which implies 2^{2^n} is not $O(2^n)$.

21. Arrange the functions $\sqrt{n}$, $1000 \log n$, $n \log n$, $2n!$, 2^n, 3^n, and $n^2/1{,}000{,}000$ in a list so that each function is big-O of the next function.

Sol: $1000 \log n$, $\sqrt{n}$, $n \log n$, $\frac{n^2}{1000000}$, 2^n, 3^n, $2n!$

23. Suppose that you have two different algorithms for solving a problem. To solve a problem of size n, the first algorithm uses exactly $n(\log n)$ operations and the second algorithm uses exactly $n^{3/2}$ operations. As n grows, which algorithm uses fewer operations?

Since $\log n < n^{\frac{1}{2}}$, then $n \log n < n^{\frac{3}{2}}$.

Thus, when n gets larger, the first algorithm uses fewer operations than the second one.

25. Give as good a big-O estimate as possible for each of these functions.

a) $(n^2 + 8)(n + 1)$ **b)** $(n \log n + n^2)(n^3 + 2)$

c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

Sol a) $|(n^2+8)(n+1)| = |n^3+n^2+8n+8| < |n^3+n^3+8n^3+8n^3| < 18|n^3|$

$\Rightarrow (n^2+8)(n+1)$ is $O(n^3)$

b) $|(n\log n + n^2)(n^3+2)| < |(n^2+n^2)(n^3+2)| < |2n^2 \cdot 2n^3| < 4|n^5|$

$\Rightarrow (n\log n + n^2)(n^3+2)$ is $O(n^5)$

c) $|(n!+2^n)(n^3+\log(n^2+1))| < |(n!+n!)(n^3+n^3)| < 4|n!n^3|$

$\Rightarrow (n!+2^n)(n^3+\log(n^2+1))$ is $O(n!n^3)$

27. Give a big-O estimate for each of these functions. For the function g in your estimate that $f(x)$ is $O(g(x))$, use a simple function g of the smallest order.

a) $n\log(n^2+1)+n^2\log n$

b) $(n\log n+1)^2+(\log n+1)(n^2+1)$

c) $n^{2^n}+n^{n^2}$

Sol

a) Let $f(n)=n\log(n^2+1)+n^2\log n$, we have

$$|f(n)|<|n\log(n^2+n^2)+n^2\log n|<|n\log(2n^2)+n^2\log n|$$

$$<|n\cdot\log 2+n\log n^2+n^2\log n|$$

$$<|n\log 2+2n\log n+n^2\log n|$$

$$<|n^2\log n+2n^2\log n+n^2\log n|<4|n^2\log n|$$

$\Rightarrow$ $f(n)$ is $O(n^2\log n)$

b) Let $f(n)=(n\log n+1)^2+(\log n+1)(n^2+1)$

$$=n^2(\log n)^2+2n\log n+1+n^2\log n+n^2+\log n+1$$

$$|f(n)|\leq|n^2(\log n)^2+2n^2(\log n)^2+n^2(\log n)^2+n^2(\log n)^2+n^2(\log n)^2+n^2(\log n)^2+n^2(\log n)^2|$$

$$\leq 8|n^2(\log n)^2|$$

$\Rightarrow$ $f(n)$ is $O(n^2(\log n)^2)$

c) Let $f(n)=2^{n^2}+2^{2^n}$

Since 2^x is strictly increasing and $n^2<2^n$ for $n>4$.

Then $2^{n^2}<2^{2^n}$ for $n>4$

$$\Rightarrow |f(n)|\leq|2^{n^2}+2^{2^n}|<|2^{2^n}+2^{2^n}|<2|2^{2^n}|$$

$\Rightarrow$ $f(n)$ is $O(2^{2^n})$

34. **a)** Show that $3x^2 + x + 1$ is $\Theta(3x^2)$ by directly finding the constants k, C_1, and C_2 in Exercise 33.

b) Express the relationship in part (a) using a picture showing the functions $3x^2 + x + 1$, $C_1 \cdot 3x^2$, and $C_2 \cdot 3x^2$, and the constant k on the x-axis, where C_1, C_2, and k are the constants you found in part (a) to show that $3x^2 + x + 1$ is $\Theta(3x^2)$.

Sol. Let $f(x) = 3x^2+x+1$.

a) ① To find a big-O estimate of $f(x)$, we have

$|f(x)| \leq |3x^2+x+1| < |3x^2+x^2+x^2| < 5|x^2|$, $x>1$

$\Rightarrow f(x)$ is $O(x^2)$ with $C_1=5$, $k=1$.

② To find a big-Ω estimate of $f(x)$, we have

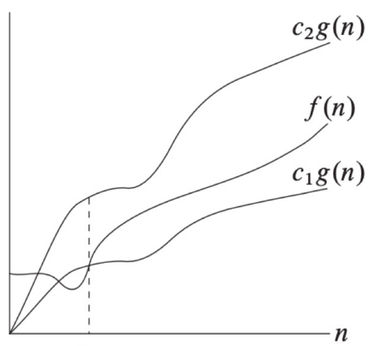
$|f(x)| \geq |3x^2+x+1| > |3x^2| > 3|x^2|$, $x>1$

$\Rightarrow f(x)$ is $\Omega(x^2)$ with $C_2=3$, $k=1$

By ①, ②, $f(x)$ is $\Theta(x^2)$

b)

35. Express the relationship $f(x)$ is $\Theta(g(x))$ using a picture. Show the graphs of the functions $f(x)$, $C_1|g(x)|$, and $C_2|g(x)|$, as well as the constant k on the x-axis.



$f(n) = \Theta(g(n))$

36. Explain what it means for a function to be $\Omega(1)$.

Let $f(n)$ be $\Omega(1)$. It means that

$|f(n)| > C|1|$ for a sufficiently large n.

It implies that $f(n)$ has a lower bound.

37. Explain what it means for a function to be $\Theta(1)$.

Let $f(n)$ be $\Theta(1)$. It means that there exists C_1, C_2 such that $C_2|1| < |f(n)| < C_1|1|$ for a sufficiently large n.

Thus, $f(n)$ is bounded betwen C_1, and C_2.

38. Give a big-O estimate of the product of the first n odd positive integers.

Let $f(n) = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots (2n-1)$.

We have $|f(n)| < |2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots 2n| < |2^n \cdot n!|$, $n > 1$

$\Rightarrow f(n)$ is $O(2^n \cdot n!)$ when $C=1$, $k=1$.

39. Show that if f and g are real-valued functions such that $f(x)$ is $O(g(x))$, then for every positive integer n, $f^n(x)$ is $O(g^n(x))$. [Note that $f^n(x) = f(x)^n$.]

Since $f(x)$ is $O(g(x))$, it means that there exists C, K such that $|f(x)| < C|g(x)|$ for $x > K$, then

$$\underbrace{|f(x)| \cdot |f(x)| \cdots |f(x)|}_{n \text{ times}} < \underbrace{C|g(x)| \cdot C|g(x)| \cdots C|g(x)|}_{n \text{ times}}$$

$\Rightarrow |f^n(x)| < C^n |g^n(x)|$ for $x > K$

It means that $f^n(x)$ is $O(g^n(x))$ when C and K.

40. Show that for all real numbers a and b with $a > 1$ and $b > 1$, if $f(x)$ is $O(\log_b x)$, then $f(x)$ is $O(\log_a x)$.

Since $\log_b x = \frac{\ln x}{\ln b}$ and $\log_a x = \frac{\ln x}{\ln a}$ where $\ln b$, $\ln a$ are two constants.

Then, if $f(x)$ is $O(\log_b x)$, we have

$$|f(x)| < c\,|\log_b x| \text{ for } x > k.$$

$$\leq c\left|\frac{\ln x}{\ln b}\right| \leq \frac{\ln b}{\ln a}\bar{c}\left|\frac{\ln x}{\ln b}\right| \leq \bar{c}\left|\frac{\ln x}{\ln a}\right| \leq \bar{c}\,|\log_a x|$$

which implies $f(x)$ is $O(\log_a x)$.

41. Suppose that $f(x)$ is $O(g(x))$, where f and g are increasing and unbounded functions. Show that $\log |f(x)|$ is $O(\log |g(x)|)$.

Since $\log x$ is strictly increasing, then $1 < x_1 < x_2$ implies $\log x_1 < \log x_2$

Since $f(x)$ is $O(g(x))$, it means there exists c, k such that

$$|f(x)| < c\,|g(x)| \text{ for } x > k.$$

Then, putting $|f(x)|$, $c|g(x)|$ in $\log x$, we have

$$\log(|f(x)|) < \log(c\,|g(x)|) = \underbrace{\log c}_{\text{a constant}} + \log|g(x)|$$

$$< \bar{c}\cdot\log|g(x)|, \quad \bar{c} = \log c + 1.$$

$\Rightarrow \log|f(x)|$ is $O(\log|g(x)|)$

42. Suppose that $f(x)$ is $O(g(x))$. Does it follow that $2^{f(x)}$ is $O(2^{g(x)})$?

NO. For example, let $f(x)=2x$, we have

$|f(x)| < 2|x|$, $x>1 \Rightarrow f(x)$ is $O(x)$ so $g(x)=x$.

Then $2^{f(x)} = 2^{2x} = (2^2)^x = 4^x$ and $2^{g(x)} = 2^x$, we have

$4^x > 2^x$ for $x>1$, and 4^x is not $O(2^x)$

$\Rightarrow$ $2^{f(x)}$ is not $O(2^{g(x)})$.