Section 2.3

3. Determine whether f is a function from the set of all bit strings to the set of integers if

a) $f(S)$ is the position of a 0 bit in S.
b) $f(S)$ is the number of 1 bits in S.
c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

Sol a) NOT a function.
For those bit strings S with multiple zeros, $f(S)$ have more than one value, which means one input gets multiple outputs, therefore $f(S)$ is NOT a function

b) It's a function.
For each bit string S, the number of 1 bits in S is a fixed and unique number. Thus, $f(S)$ is a function

c) Not a function.

9. Find these values.

a) $\lceil \frac{3}{4} \rceil = 1$
b) $\lfloor \frac{7}{8} \rfloor = 0$
c) $\lceil -\frac{3}{4} \rceil = 0$
d) $\lfloor -\frac{7}{8} \rfloor = -1$
e) $\lceil 3 \rceil = 3$
f) $\lfloor -1 \rfloor = -1$
g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rfloor = 2$
h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot 2 \rfloor = 1$

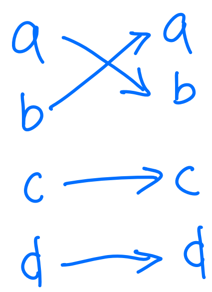
10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

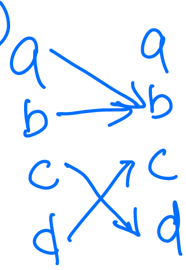
c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

(a)



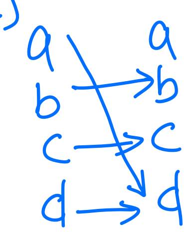
Yes, different input gets different output

(b)



NO. $a \neq b$ but $f(a) = f(b)$

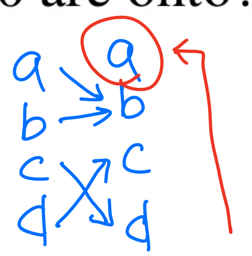
(c)



NO, $a \neq d$ but $f(a) = f(d)$.

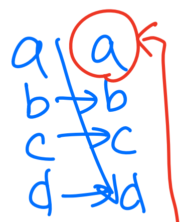
11. Which functions in Exercise 10 are onto?

Only (a) is onto. In (b)



has no input

and (c)



has no input.

12. Determine whether each of these functions from $\mathbf{Z}$ to $\mathbf{Z}$ is one-to-one.

a) $f(n) = n - 1$ **b)** $f(n) = n^2 + 1$

c) $f(n) = n^3$ **d)** $f(n) = \lceil n/2 \rceil$

<u>Sol</u> a). Yes, if $f(a) = f(b)$ for $a, b \in \mathbb{Z}$, then $a-1 = b-1 \Rightarrow a = b$

b) NO, let $a = 1, b = -1$, we have $f(a) = 1^2 + 1 = 2$ and $f(b) = (-1)^2 + 1 = 2$

Since $a \neq b$ and $f(a) = f(b)$, then f is NOT 1-1

c) Yes, if $f(a) = f(b)$ for $a, b \in \mathbb{Z}$, then $a^3 = b^3 \Rightarrow \sqrt[3]{a^3} = \sqrt[3]{b^3} \Rightarrow a = b$.

d) NO, let $a=2$, $b=3$, we have $f(a)=\lceil \frac{2}{2} \rceil = 1$ and $f(b)=\lceil \frac{3}{2} \rceil = 1$. since $a \neq b$ and $f(a)=f(b)$, then f is NOT 1-1

15. Determine whether the function $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.

b) $f(m, n) = m^2 + n^2$.

c) $f(m, n) = m$.

d) $f(m, n) = |n|$.

e) $f(m, n) = m - n$.

<u>Sol</u>. a) Yes, any integer can be represented by the addition of the other two integers

b) NO, since $m^2+n^2 \geq 0$, then if the output is a negative integer, there is NO such $m, n \in \mathbb{Z}$ that $m^2+n^2 =$ a negative integer.

c) Yes. for each output $m \in \mathbb{Z}$, there is an input $m \in \mathbb{Z}$ such that $f(m,n) = m$

d) No. since $f(m,n) = |n| \geq 0$, then there is no negative integer as an output. which means those negative integers have no input.

e) Yes, any integer can be represented by a subtraction of two integers.

20. Give an example of a function from **N** to **N** that is

a) one-to-one but not onto.
b) onto but not one-to-one.
c) both onto and one-to-one (but different from the identity function).
d) neither one-to-one nor onto.

Sol: (a) $f(x) = x^2$

One-to-one: $f(a) = f(b) \Rightarrow a^2 = b^2 \Rightarrow a = b$
(since $a, b \in \mathbb{N}$)
so $a \neq -b$

NOT onto, if $f(x) = 3 \Rightarrow x = \sqrt{3} \notin \mathbb{N}$.

(b) $f(x) = \lceil \frac{x}{2} \rceil$

NOT one-to-one:
Let $a = 1$, $b = 2$ and $f(a) = \lceil \frac{1}{2} \rceil = 1$
$f(b) = \lceil \frac{2}{2} \rceil = 1$

onto: for each output $\in \mathbb{N}$, there is an input $\in \mathbb{N}$.

(c) $f(x) = \lceil x \rceil$
or $f(x) = \lfloor x \rfloor$

(identity function is $f(x) = x$)

(d) $f(x) = (x-3)^2$

NOT 1-1: $x = 4 \Rightarrow f(4) = 1$
$x = 2 \Rightarrow f(2) = 1$
$4 \neq 2$ but $f(2) = f(4)$.

NOT onto $f(x) = 2 \Rightarrow (x-3)^2 = 2$
$\in \mathbb{N}$
$\Rightarrow x = 3 \pm \sqrt{2} \notin \mathbb{N}$.

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$
d) $f(x) = (x^2 + 1)/(x^2 + 2)$

Sol: (a) $f(x)=2x+1$.

① Prove f is 1-1: if $f(a)=f(b)$, $\Rightarrow 2a+1=2b+1$
$\Rightarrow 2a=2b \Rightarrow a=b$, then f is 1-1.

② Prove f is onto: let $y=2x+1$, for all $y\in\mathbb{R}$,
we have $x=\frac{y-1}{2}\in\mathbb{R}$, then f is onto.

$\boxed{\text{By ①,②, } f \text{ is a bijection.}}$

(b) $f(x)=x^2+1$.

① NOT one-to-one: let $a=1, b=-1$, then $f(a)=1+1=2$, $f(b)=(-1)^2+1=2$

It means $a\neq b$ but $f(a)=f(b)$, then f is NOT 1-1.

② NOT onto: counterexample, if output $=0$, we have
$x^2+1=0 \Rightarrow x^2=-1$, $x=\pm i \notin \mathbb{R}$

$\boxed{f \text{ is NOT a bijection}}$

(c) $f(x)=x^3$

① prove $f(x)$ is one-to-one:
Let $f(a)=f(b)$, we have $a^3=b^3 \Rightarrow \sqrt[3]{a^3}=\sqrt[3]{b^3}$
$\Rightarrow a=b$.

② prove $f(x)$ is onto:
Let $y=x^3$. For all $y\in\mathbb{R}$, we have $x=\sqrt[3]{y}\in\mathbb{R}$

$\boxed{\text{By ① ②, } f(x) \text{ is a bijection}}$.

(d) $f(x)=\frac{x^2+1}{x^2+2}$

① NOT one-to-one
let $a=1, b=-1$, $f(a)=\frac{1+1}{1+2}=\frac{2}{3}$, $f(b)=\frac{(-1)^2+1}{(-1)^2+2}=\frac{2}{3}$
$a\neq b$ but $f(a)=f(b)$

② NOT onto; For output $=0$, we have $\frac{x^2+1}{x^2+2}=0$

$\Rightarrow x^2+1=0 \Rightarrow x=\pm i \notin \mathbb{R}.$

$\boxed{f \text{ is NOT a bijection}}$

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$.

b) $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$.

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

Sol: a) $f(-1)=1,$

$f(0)=1$

$f(2)=1$

$f(4)=1$

$f(7)=1$

b) $f(x)=2x+1.$

$f(-1)=2\cdot(-1)+1=-1$

$f(0)=2\cdot(0)+1=1$

$f(2)=2\cdot 2+1=5$

$f(4)=2\cdot 4+1=9$

$f(7)=2\cdot 7+1=15$

c) $f(x)=\lceil \frac{x}{5} \rceil$

$f(-1)=\lceil \frac{-1}{5} \rceil=0$

$f(0)=\lceil \frac{0}{5} \rceil=0$

$f(2)=\lceil \frac{2}{5} \rceil=1$

$f(4)=\lceil \frac{4}{5} \rceil=1$

$f(7)=\lceil \frac{7}{5} \rceil=2$

d) $f(x)=\lfloor \frac{(x^2+1)}{3} \rfloor$

$f(-1)=\lfloor \frac{(-1)^2+1}{3} \rfloor=\lfloor \frac{2}{3} \rfloor=0$

$f(0)=\lfloor \frac{0+1}{3} \rfloor=\lfloor \frac{1}{3} \rfloor=0$

$f(2)=\lfloor \frac{2^2+1}{3} \rfloor=\lfloor \frac{5}{3} \rfloor=1$

$f(4)=\lfloor \frac{16+1}{3} \rfloor=\lfloor \frac{17}{3} \rfloor=5$

$f(7)=\lfloor \frac{49+1}{3} \rfloor=\lfloor \frac{50}{3} \rfloor=16$

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Proof (a): "f is one-to-one" implies $f(s) = f(t) \Rightarrow s = t$

"g is one-to-one" implies $g(a) = g(b) \Rightarrow a = b$

To show $f \circ g$ is one to one, let $(f \circ g)(a) = (f \circ g)(b)$

$\Rightarrow f(g(a)) = f(g(b)) \xRightarrow{\text{since } f \text{ is } 1\text{-}1} g(a) = g(b) \xRightarrow{\text{since } g \text{ is } 1\text{-}1} a = b.$

It means $(f \circ g)(a) = (f \circ g)(b)$ implies $a = b$.

Therefore $f \circ g$ is one-to-one

Proof (b): "f is onto" : For each element c in C, there is a $b \in B$ such that $f(b) = c$

$f: B \to C$

"g is onto" : For each element b in B, there is a $a \in A$ such that $g(a) = b$

$g: A \to B$

To show $f \circ g$ is onto; for each $c \in C$, since
($f(g(x))$)

f is onto, there exist $g(x) = b$ such that $f(g(x)) = c$.

Since g is onto, for each b, there exists a $a \in A$ such that $g(a) = b$

$\Rightarrow$ for each $c \in C$, there exists a $a \in A$ such that

$f(g(a)) = c. \Rightarrow f(g(x))$ is onto.

41. Show that the function $f(x) = ax + b$ from $\mathbf{R}$ to $\mathbf{R}$, where a and b are constants with $a \neq 0$ is invertible, and find the inverse of f.

Sol: ① To show $f(x) = ax+b$ from $\mathbb{R}$ to $\mathbb{R}$ with $a \neq 0$ is invertible.

We need show $f(x)$ is bijective.

(1) To show $f(x)$ is one-to-one, Let $f(s) = f(t)$.

$\Rightarrow as+b = at+b \Rightarrow as = at \underset{a\neq 0}{\Rightarrow} s = t$

Which means f is 1-1

(2) To show $f(x)$ is onto. Let $y = ax+b$.

For all $y \in \mathbb{R}$, we have $ax = y-b \Rightarrow x = \frac{y-b}{a} \in \mathbb{R}$

Thus, f is onto.

By (1), (2) f is bijective $\Rightarrow$ f is invertible.

② Find f^{-1}

(1) Let $f(x) = y$: $y = ax+b$

(2) swith x & y : $x = ay+b$

(3) solve for y: $ay = x-b \Rightarrow y = \frac{x-b}{a}$

(4) replace y by $f^{-1}(x)$: $f^{-1}(x) = \frac{x-b}{a}$.

Let f be a function from the set A to the set B. Let S be a subset of B. We define the **inverse image** of S to be the subset of A whose elements are precisely all preimages of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so $\boxed{f^{-1}(S) = \{a \in A \mid f(a) \in S\}}$. [*Beware:* The notation f^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $f^{-1}(y)$ for the value at y of the inverse of the invertible function f. Notice also that $f^{-1}(S)$, the inverse image of the set S, makes sense for all functions f, not just invertible functions.]

44. Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$.

b) $f^{-1}(\{x \mid 0 < x < 1\})$.

c) $f^{-1}(\{x \mid x > 4\})$.

Sol: a) $f^{-1}(\{1\}) = \{-1, 1\}$

b) $f^{-1}(\{x \mid 0 < x < 1\}) = \{a \mid a \in (-1,0) \cup (0,1)\}$

c) $f^{-1}(\{x \mid x > 4\}) = \{a \mid a \in (-\infty, -2) \cup (2, \infty)\}$

45. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\})$.

b) $g^{-1}(\{-1, 0, 1\})$.

c) $g^{-1}(\{x \mid 0 < x < 1\})$.

Sol a) $g^{-1}(\{0\}) = \{a \mid 0 \le a < 1\}$

b) $g^{-1}(\{-1, 0, 1\}) = \{a \mid -1 \le a < 2\}$

c) $g^{-1}(\{x \mid 0 < x < 1\}) = \phi$

46. Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Sol: a) Since " $f^{-1}(S \cup T)$ " and " $f^{-1}(S) \cup f^{-1}(T)$ " are sets, then, to show (a) is true, we need to prove:

① $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and ② $f^{-1}(S \cup T) \supseteq f^{-1}(S) \cup f^{-1}(T)$

Proof ①: Let $x \in f^{-1}(S \cup T)$, we have

$f(x) \in S \cup T$

$\Rightarrow f(x) \in S$ or $f(x) \in T$

$\Rightarrow x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\Rightarrow x \in f^{-1}(S) \cup f^{-1}(T)$

Thus, $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$

Proof ②: Let $x \in f^{-1}(S) \cup f^{-1}(T)$, we have $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\Rightarrow f(x) \in S$ or $f(x) \in T$

$\Rightarrow f(x) \in S \cup T$

$\Rightarrow x \in f^{-1}(S \cup T)$

Thus, $f^{-1}(S \cup T) \supseteq f^{-1}(S) \cup f^{-1}(T)$

By ①, ②, we have $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

b) To show $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, we need to show

① $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and ② $f^{-1}(S \cap T) \supseteq f^{-1}(S) \cap f^{-1}(T)$

Proof ①: Let $x \in f^{-1}(S \cap T)$, we have

$f(x) \in S \cap T$

$\Rightarrow f(x) \in S$ and $f(x) \in T$

$\Rightarrow x \in f^{-1}(S)$ and $x \in f^{-1}(T)$

$\Rightarrow x \in f^{-1}(S) \cap f^{-1}(T)$

Thus, $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$

Proof ②: Let $x \in f^{-1}(S) \cap f^{-1}(T)$. We have $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$

$\Rightarrow f(x) \in S$ and $f(x) \in T$

$\Rightarrow f(x) \in S \cap T$

$\Rightarrow x \in f^{-1}(S \cap T)$

Thus, $f^{-1}(S \cap T) \supseteq f^{-1}(S) \cap f^{-1}(T)$

By ①, ②, we have $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

60. How many bytes are required to encode n bits of data where n equals

a) 4? **b)** 10? **c)** 500? **d)** 3000?

In example 29 (p158), each byte is made up of 8 bits.

a) if $n=4$, we need $\lceil \frac{4}{8} \rceil = 1$ byte

b) if $n=10$, we need $\lceil \frac{10}{8} \rceil = 2$ bytes

c) if $n=500$, we need $\lceil \frac{500}{8} \rceil = \lceil 62.5 \rceil = 63$ bytes

d) if $n=3000$, we need $\lceil \frac{3000}{8} \rceil = \lceil 375 \rceil = 375$ bytes.

61. How many bytes are required to encode n bits of data where n equals

a) 7? **b)** 17? **c)** 1001? **d)** 28,800?

a) if $n=7$, we need $\lceil \frac{7}{8} \rceil = 1$ byte.

b) if $n=17$, we need $\lceil \frac{17}{8} \rceil = \lceil 2\frac{1}{8} \rceil = 3$ bytes

c) if $n=1001$, we need $\lceil \frac{1001}{8} \rceil = \lceil 125\frac{1}{8} \rceil = 126$ bytes

d) if $n=28800$, we need $\lceil \frac{28800}{8} \rceil = \lceil 3600 \rceil = 3600$ bytes

62. How many ATM cells (described in Example 30) can be transmitted in 10 seconds over a link operating at the following rates?

a) 128 kilobits per second (1 kilobit = 1000 bits)
b) 300 kilobits per second
c) 1 megabit per second (1 megabit = 1,000,000 bits)

Sol : "53 bytes per cell" $\Rightarrow$ "$53 \cdot 8 = 424$ bits per cell"

a) 128 000 bits per second × 10 second = 1280000 bits

$$\left\lfloor \frac{1280000}{424} \right\rfloor = \lfloor 3018.86\cdots \rfloor = 3018 \text{ cells}$$

b) 300 000 bits per sec. × 10 sec. = 3000000

$$\left\lfloor \frac{3000000}{424} \right\rfloor = \lfloor 7075.47\cdots \rfloor = 7075 \text{ cells}$$

c) 1000000 bits per sec. × 10 sec. = 10000000

$$\left\lfloor \frac{10000000}{424} \right\rfloor = \lfloor 23584.90\cdots \rfloor = 23584 \text{ cells}$$

63. Data are transmitted over a particular Ethernet network in blocks of 1500 octets (blocks of 8 bits). How many blocks are required to transmit the following amounts of data over this Ethernet network? (Note that a byte is a synonym for an octet, a kilobyte is 1000 bytes, and a megabyte is 1,000,000 bytes.)

a) 150 kilobytes of data $\rightarrow \left\lceil \frac{150000 \text{ bytes}}{1500} \right\rceil = 100$ blocks
b) 384 kilobytes of data $\rightarrow \left\lceil \frac{384000}{1500} \right\rceil = 256$ blocks
c) 1.544 megabytes of data
d) 45.3 megabytes of data

c) $\left\lceil \dfrac{1544000}{1500} \right\rceil = \lceil 1029.333\ldots \rceil = 1030$ blocks

d) $\left\lceil \dfrac{45300000}{1500} \right\rceil = \lceil 30200 \rceil = 30200$ blocks.