MAT2440, Classwork46, Spring2025

ID:_____ Name:_____

II. Examples of Mathematical Induction.

1. Prove that $n^2 + n$ is divisible by 2 for all $n \in \mathbb{Z}^+$. → P(n)

Proof ① show P(1) is true:

$(1)^2+1$ is divisible by 2 True (2 is divisible by 2)

② Assume P(k) is true.

P(k): k^2+k is divisible by 2 $\Leftrightarrow \frac{k^2+k}{2} = m \Leftrightarrow k^2+k=2m$

③ Prove P(k) → P(k+1)

P(k+1) looks like: $(k+1)^2+(k+1)$ is divisible by 2

$L.H.S = (k+1)^2+(k+1) = k^2+2k+1+k+1$

$= k^2+k+2k+1+1$

$= 2m+2k+2$ (the assumption in ②)

$= 2(m+k+1)$

$\Rightarrow (k+1)^2+(k+1)$ is divisible by 2

The statement is true for all $n \in \mathbb{Z}^+$ by induction

2. Prove that $n^3 + 2n$ is divisible by 3 for every integer $n \geq 3$. P(n)

Proof: ① show P(3) is true

$3^3+2\cdot 3 = 27+6 = 33$ is divisible by 3 True.

② Assume P(k) is true

k^3+2k is divisible by 3 $\Leftrightarrow k^3+2k = 3m$, $m \in \mathbb{Z}$.

③ Prove P(k) → P(k+1)

P(k+1) looks like $(k+1)^3+2(k+1)$ is divisible by 3

$L.H.S = (k+1)^3+2(k+1)$

$= k^3+3k^2+3k+1+2k+2$

$= k^3+2k+3k^2+3k+1+2$

$= 3m+3k^2+3k+3$ (based on the assumption in ②)

$= 3(m+k^2+k+1) \Rightarrow$ This is divisible by 3

By induction, based on ①, ②, ③, this statement is true for $n \geq 3$

3. Use mathematical induction to prove the inequality for all positive integer n

$n < 2^n$. $p(n)$ $n \in \mathbb{Z}^+$

Proof: ① show P(1) is true

$1 < 2^1$ True

② Assume P(k) is true.: $k < 2^k$ is true. → P(k)

③ Prove P(k) → P(k+1)

P(k+1) looks like

$k+1 < 2^{k+1}$

(Left Hand side)

L.H.S = $k+1$

based on the assumption in ② $< 2^k + 1$

$< 2^k + 2^k$ ↘ $1 < 2^k$, for $k \in \mathbb{Z}^+$

$= 2 \cdot 2^k = 2^{k+1}$ = R.H.S

By induction, based ①, ②, ③, this statement is true for all $n \in \mathbb{Z}^+$

4. Use mathematical induction to prove the inequality for every positive integer $n \geq 4$

$2^n < n!$