MAT2440, Classwork45, Spring2025

ID:________________________________ Name:________________________________

I. Introduction of Mathematical Induction

1. Examples of mathematical statements assert that a property is true for all positive integers.

- $n! < n^n$ for all $n \in \mathbb{Z}^+$
- $n^3 - n$ is divisible by 3 for all $n \in \mathbb{Z}^+$
- A set of n elements has 2^n subsets for all $n \in \mathbb{Z}^+$
- $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$

A major goal of this chapter is to provide a thorough understanding of mathematical induction, which is used to prove results of this kind of statement.

2. Find some terms of the given recursive sequence

$$a_1 = 1,\ a_n = a_{n-1} + 2n$$

$a_1 = 1 \longleftarrow$ initial value

$a_2 = a_1 + 2(2) = 1 + 4 = 5$

$a_3 = a_2 + 2\cdot(3) = 5 + 6 = 11$

$a_4 = a_3 + 2\cdot(4) = 11 + 8 = 19$

find a_n from the value of a previous term a_{n-1}

3. The Principle of Mathematical Induction.

Now, let $P(n)$ be a propositional function that we want to prove for All $n \in \mathbb{Z}^+$

Mathematical Induction:

① Prove $P(1)$ is true, i.e. the statement is true for $n=1$

② Assume $P(k)$ is true

③ Prove $P(k) \to P(k+1)$.

i.e. if the statement $P(k)$ is true, then it shows that $P(k+1)$ is true as well.

The above 3 steps allow you to say $P(n)$ is true for ALL $n \in \mathbb{Z}^+$.

In the Rule of Inference Modus ponens

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

$P(1) \to P(2) \to P(3) \cdots\cdots \Rightarrow P(n)$ is true for all $n \in \mathbb{Z}^+$

$P(1)$ true, $P(2)$ true, $P(3)$ true

II. Examples of Mathematical Induction. $\rightarrow P(n)$

4. Prove that $\boxed{1+2+3+\cdots+n=\frac{n(n+1)}{2}}$ for all $n \in \mathbb{Z}^+$.

Proof ① Show $P(1)$ is true (i.e. this statement is true when $n=1$)

$$1=\frac{1\cdot(1+1)}{2} \Rightarrow 1=1 \quad \text{True}$$

② Assume $P(k)$ is true

$$1+2+3+\cdots+k=\frac{k(k+1)}{2} \text{ is true}$$

③ Prove $P(k) \rightarrow P(k+1)$ (i.e. prove $P(k+1)$ is true based on $P(k)$ is true)

The left hand side of $P(k+1) = \underbrace{1+2+3+\cdots+k}+(k+1)$

based on the assumption in ② $= \frac{k(k+1)}{2} + (k+1)$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2}$$

$(k+1)$ is a common factor

$$=\frac{(k+1)(k+2)}{2}=\frac{(k+1)(k+1+1)}{2} = \text{right hand side of } P(k+1)$$

$\Rightarrow$ The statement is true for all $n \in \mathbb{Z}^+$ by induction.

5. Prove that $\boxed{1+3+5+7+\cdots+(2n-1)=n^2}$ for all $n \in \mathbb{Z}^+$. $\rightarrow P(n)$

proof: ① show $P(1)$ is true:

(that is, the statement is true when $n=1$)

$$1=1^2 \quad \text{True}$$

② Assume $P(k)$ is true

$$1+3+5+7+\cdots+(2k-1)=k^2 \quad \text{True}$$

③ Prove $P(k) \rightarrow P(k+1)$ (that is. prove $p(k+1)$ is true based on $p(k)$ is true)

$P(k+1)$ looks like

$$1+3+5+7+\cdots+(2k-1)+(2k+1) = (k+1)^2$$

L.H.S. of $P(k+1) = \boxed{1+3+5+7+\cdots+(2k-1)}+(2k+1)$

k^2 (based on the assumption in ②)

$$=k^2+(2k+1)$$

$$=k^2+2k+1=(k+1)(k+1)=(k+1)^2$$

$\Rightarrow$ the statement is true for all $n \in \mathbb{Z}^+$ by induction.