MAT1372, Classwork13, Fall2025

4.2 Bernoulli Distribution

1. Bernoulli Random Variable.

   How to describe a Bernoulli random variable in word? It has only 2 outcomes

   Example: The bar exam is a pass/ fail exam with probability of a pass as p and a fail as $1-p$

   How to describe a Bernoulli random variable in math? set one outcome to be 1 and the other be 0.

   Example: In bar exam, set a pass be 1 and a fail be 0.

2. The Mean and Standard Deviation of a Bernoulli Random Variable.

If X is a random variable that takes value 1 with probability of success p and 0 with probability $1-p$, then X follows Bernoulli distribution with

$$E(X) = \mu = \underline{p} \text{ and } \sigma = \underline{\sqrt{p(1-p)}}$$

4.3 Binomial Distribution

1. In the example of the bar exam, assume the probability of a pass $p = 0.7$. If four individuals A, B, C, and D took this exam, what is the chance exactly one of them will fail the exam?

$P(\text{exactly one fails}) = 4 \cdot (0.3) \cdot (0.7)^3 = 4 \cdot (0.103) = 0.412$

$P(A=\text{fail}, BCD=\text{pass}) = (0.3)(0.7)^3$

$= {}_4C_1\,(0.3)\cdot(0.7)^3 = \binom{4}{1}(0.3)(0.7)^3$

$$\boxed{\binom{4}{1} = \frac{4!}{1!\,(4-1)!} = \frac{4!}{3!} = 4}$$

2. Introduction of the Binomial Distribution.

   How to describe a Binomial Distribution in word?

It is used to describe the number of successes in a fixed number of trails.

   How to describe a Binomial Distribution in math?

It describes the probability of having exactly k successes in n independent Bernoulli trail with propbability of a success as p.

   Example: In 1 (the bar exam) we have $n=4$, $k=3$, $p=0.7$. The final probability [the number of scenarios] × P(single scenario)

3. Definition of the Binomial Distribution.

Suppose the probability of a single trail being a success is p. Then the probability of observing exactly k successes in n independent trails is given by

$$P(\text{exactly } k \text{ successes in } n) = \binom{n}{k} p^k \cdot (1-p)^{n-k} = \frac{n!}{k!\,(n-k)!}\, p^k \cdot (1-p)^{n-k}$$

$$\mu = \underline{np},\ \sigma^2 = \underline{np(1-p)},\ \sigma = \underline{\sqrt{np(1-p)}}$$

(Assume $X_1, X_2, X_3, \ldots, X_n$ are Bernolli R.V.
Then $X = X_1 + X_2 + X_3 + \cdots + X_n$ follows Binomial Distribution)

4. Is it Binomial? Four conditions to check:

(1) The trails are independent

(2) The number of trails, n, is fixed

(3) Each trail's outcome can be classified as a success or a failure

(4) The probability of a success, p, is the same in each trail.

5. Computing Binomial Probabilities.

The first step in using the binomial model is to check that the model is appropiate.

The second step is to identify n, k, p.

As the last stage use the formulas to determine the probability, then interpret the results

6. In the bar exam with $p = 0.7$, What is the probability that 3 of 8 randomly selected individuals will have failed the exam, i.e. that 5 of 8 will pass it?

$$P(\text{exactly 5 pass in 8 individual}) = \binom{8}{5} \cdot (0.7)^5 (0.3)^3 = 0.254$$

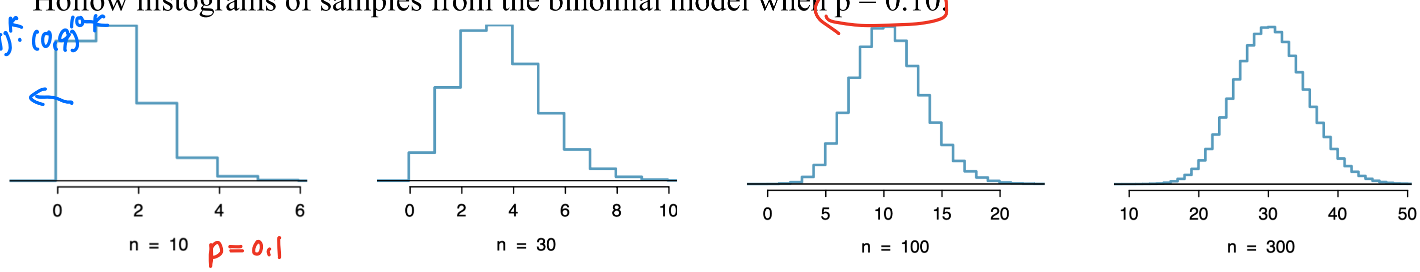
7. If we randomly sampled 40 n people who took bar exam, how many of the people would you expect to pass the exam in a given year? What is the standard deviation of the number that would pass the exam?

$$E(X) = \mu = n \cdot p = 40 \cdot 0.7 = 28 \quad , \quad \sigma = \sqrt{n \cdot p(1-p)} = \sqrt{40 \cdot 0.7 \cdot 0.3} = 2.9$$

8. Observation: the Binomial Distribution with a large sample size.

Hollow histograms of samples from the binomial model when p = 0.10.

$\binom{10}{k}(0.1)^k \cdot (0.9)^{10-k}$



The sample sizes for the four plots are n = 10, 30, 100, and 300, respectively. What do you observe?

The distribution is getting more and more symmetrical and it looks like a normal distribution.

9. Normal Approximation of the Binomial Distribution.

The binomial distribution with probability of success p is nearly normal when the sample size n is sufficiently large such that

① $n \cdot p > 10$. The approximate normal distribution has

② $n(1-p) > 10$

the mean $\mu = np$, $\sigma = \sqrt{np(1-p)}$ $\quad (X \sim N(np, \sqrt{np(1-p)}))$

10. Given a random variable X and it follows the Binomial Distribution with $n = 400$ and $p = 0.15$.

(a) Find the mean μ and standard deviation σ. $\mu = 400 \cdot 0.15 = 60$, $\sigma = \sqrt{60 \cdot 0.85} = 7.14$

(b) By calculating, we know $P(X < 42) = 0.0054$. If $Y \sim N(\mu, \sigma)$, find $P(Y < 42)$ by the table.

① $np = 60 > 10$ $\quad$ If $Y \sim N(60, 7.14)$

② $n(1-p) = 340 > 10$.

$$P(Y < 42) = P(Z < -2.52) = 0.0059$$

$$Z_1 = \frac{42-60}{7.14} = -2.52$$