MAT1375, Classwork10, Fall2025

Ch10. Rational Functions I

1. Definition of the **Rational function**:

A rational function is a fraction of two polynomials $f(x) = \frac{p(x)}{g(x)}$, where $p(x)$ and $g(x)$ are both polynomials, and $g(x) \neq 0$.

The **domain of a rational function** f is all real numbers for which the denominator $g(x)$ is not zero:

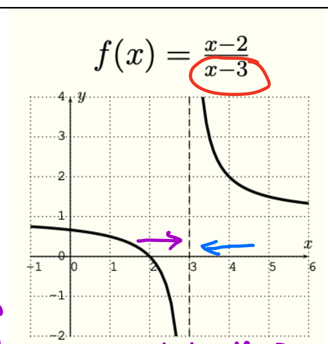
$$D_f = \{ x \in \mathbb{R} \mid D_p \cap D_g \text{ and } g(x) \neq 0 \}$$

2. **Arrow Notation**: Given a constant a and we have

$x \to a^+$:	x approaches a from the **right** (x is very closed to a but $x \neq a$ and x > a)
$x \to a^-$:	x approaches a from the **left** (x is very closed to a but $x \neq a$ and x < a)
$x \to \infty$:	x approaches infinity (x increases without bound)
$x \to -\infty$:	x approaches negative infinity (x decreases without bound)

3. The definition of a **Vertical Asymptote**:

The line $\boldsymbol{x = a}$ is a vertical asymptote of the graph of a function f if $f(x)$ **increases or decreases without bound** as $\boldsymbol{x}$ **approaches** $\boldsymbol{a}$. Here are three examples:



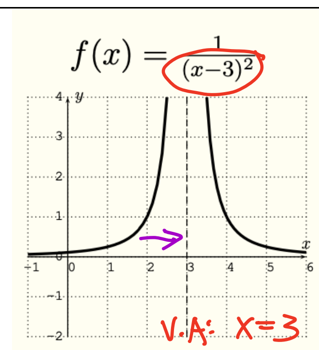
V.A.: X=3

closed to 3 from the left

As $x \to 3^-$, $f(x) \to -\infty$;

As $x \to 3^+$, $f(x) \to \infty$.

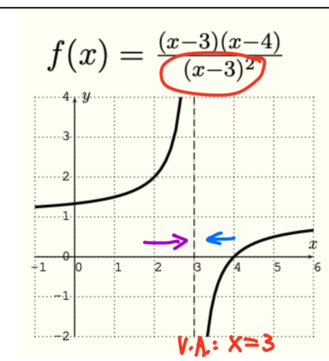
closed to 3 from the right



V.A: X=3

As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to \infty$



V.A.: X=3

As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to -\infty$

4. How to locate Vertical Asymptotes: Let $f(x) = \frac{p(x)}{g(x)}$ be a rational function.

(1) If $p(x)$ and $g(x)$ have no common factor(s), and $\boldsymbol{a}$ is a **zero** of $g(x)$ which makes $f(x)$ undefined, then $x = a$ is a vertical asymptote of the graph of $f(x)$.

(∞ or $-\infty$ or unbounded)

(2) If $\boldsymbol{a}$ is a **zero** of both $p(x)$ and $g(x)$ ($p(a) =$ 0, $g(a) =$ 0.) which means $x - a$ is the common factor of $p(x)$ and $g(x)$, then there is a jump/ removable discontinuity / singularity at $x = a$ and there is no vertical asymptote at $x = a$.

5. Find the vertical asymptotes of the graph of each rational function:

a) $f(x) = \frac{x^1}{x^2-1}$ b) $g(x) = \frac{x-1}{x^2-1}$ c) $h(x) = \frac{x-1}{x^2+1}$

Horizontal asymptote

$\deg(p(x)) = 1 < \deg(q) = 2$

H.A. $y = 0$ (x-axis)

a) $f(x) = \frac{x}{x^2-1}$, $p(x) = x$, $q(x) = x^2-1$, $f(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2-1 = 0$

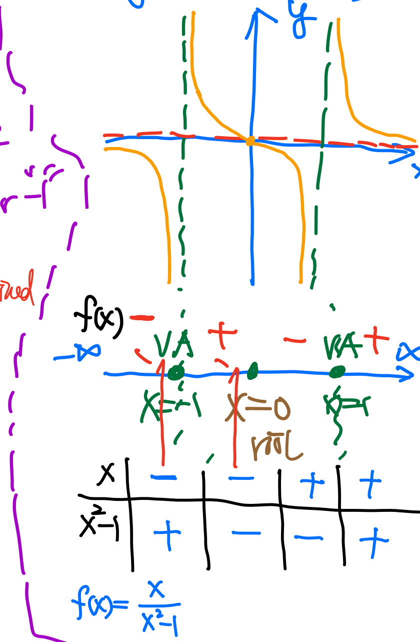
factor of -1 is "1 or -1"

$q(x) = (x+1)(x-1) = 0$

$x+1 = 0$ and $x-1 = 0$ $\Rightarrow$

$x = -1$, $x = 1$ are zeros of $q(x)$ which makes $f(x)$ undefined

(Since there is no common factor of $p(x)$ and $q(x)$),
the line $x = -1$ and $x = 1$ are vertical asymptote(s) of $f(x)$



b) $g(x) = \frac{x-1}{x^2-1}$, $p(x) = x-1$, $q(x) = x^2-1$, $g(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2-1 = 0$

$q(x) = (x-1)(x+1) = 0 \Rightarrow x-1 = 0$ or $x+1 = 0$

$\Rightarrow x = -1$ and $x = 1$ are zeros of $q(x)$ which makes $f(x)$ undefined

Since $(x-1)$ is a common factor of $p(x)$, $q(x)$,

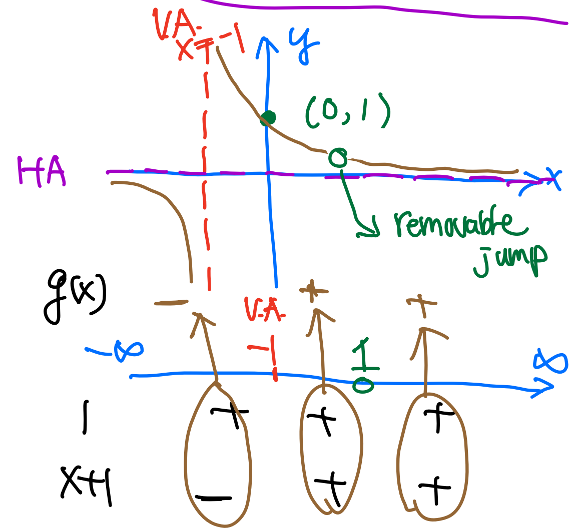
$g(x) = \frac{p(x)}{q(x)} = \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1}{x+1}$ but $x-1 \neq 0$,

then $x = 1$ is a removable discontinuity and
$x = -1$ is a vertical asymptote of $g(x)$

For H.A. $\deg(p) = 1 < \deg(q) = 2 \Rightarrow$ H.A. is $y = 0$

x-intercept: $g(x) = \frac{1}{x+1} = 0 \Rightarrow$ no x will make $g(x) = 0$
$\Rightarrow$ no x-intercept

y-intercept: $g(0) = \frac{1}{0+1} = 1 \Rightarrow (0, 1)$



c) $h(x) = \frac{x-1}{x^2+1}$, $p(x) = x-1$, $q(x) = x^2+1$, $h(x) = \frac{p(x)}{q(x)}$

To find vertical asymptote(s), let $q(x) = x^2+1 = 0$

But $q(x) = x^2+1 \geq 1$ which could never be zero

Thus, there is root of $q(x)$ which makes $h(x)$ undefined.

There is NO vertical asymptote of $h(x)$.

For H.A. $\deg(p) = 1 < \deg(q) = 2 \Rightarrow$ H.A is $y = 0$.

x-intercept: $h(x) = \frac{x-1}{x^2+1} = 0$
$\Rightarrow x = 1$. or $(1, 0)$

y-intercept: $h(0) = \frac{0-1}{0+1} = -1$
$\Rightarrow y = -1$ or $(0, -1)$