MAT1375, Classwork9, Fall2025

Ch9. Roots of Polynomials

1. Factors and roots of polynomials.

degree "n" polynomial

Every n-degree polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$, $(a_n \neq 0)$ can be factored as

$$f(x) = a_n(x-c_1)\cdot(x-c_2)\cdot(x-c_3)\cdot\cdots\cdot(x-c_n)$$

Thus, the polynomial $f(x)$ of degree n has **at most** n roots (which are $c_1, c_2, \cdots, c_n$) and these roots may be either real or complex. $(3+2i,\ \sqrt{2}+5i)$

2. The Repeat roots and its **Multiplicity.**

Let $f(x) = (x-r)^n$ where r is the root of f and this root repeats n times. We call r a root with multiplicity n.

3. The complex root and its Conjugate.

Let f be a polynomial with all **real coefficients**. The complex roots are always found as a **pair**, that is, if $c = a + bi$ is a complex root of f, then the complex conjugate $\bar{c} = a - bi$ is also a root of f.

real part imagary part

$c = 3+2i$, $\bar{c} = 3-2i$

$c = 4-i$, $\bar{c} = 4+i$

4. The Relation between Roots and Coefficient a_0.

For a n-degree polynomial $f(x) = 1x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$, we can factorize it as

$$f(x) = 1\cdot(x-c_1)\cdot(x-c_2)\cdot(x-c_3)\cdot\cdots\cdot(x-c_n).$$

where $c_1, c_2, \cdots, c_n$ are the roots of $f(x)$. Then we have

(1) $f(c_1) = 0$, $f(c_2) = 0$, $f(c_3) = 0$, ..., $f(c_n) = 0$.

(2) $a_0 = (-c_1)\times(-c_2)\times(-c_3)\times\cdots\times(-c_n)$.

(3) The factors of a_0 might be the possible candidates for the roots of $f(x)$.

5. Let $f(x) = x^3 - x^2 + 2$. Find all the roots of $f(x)$.

Step 1 guess for roots of $f(x)$: $1, -1, 2, -2$

Step2 Check the root(s)

$f(1) = 1^3 - 1^2 + 2 = 0+2 = 2 \neq 0$

$\boxed{f(-1) = (-1)^3 - (-1)^2 + 2 = -1-1+2 = 0 \checkmark}$

$f(2) = 2^3 - 2^2 + 2 = 8-4+2 = 6 \neq 0$

$f(-2) = (-2)^3 - (-2)^2 + 2 = -8-4+2 = -10 \neq 0$

$\Rightarrow (x+1)$ is a factor of f

Step3 long division $f(x)/(x+1)$

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ \hline x+1 & x^3 - x^2 + 0x + 2 \\ x^2(x+1) & -)\ x^3 + x^2 \\ \hline & -2x^2 + 0x \\ -2x(x+1) & -)\ -2x^2 - 2x \\ \hline & 2x + 2 \\ 2(x+1) & -)\ 2x + 2 \\ \hline & 0 \end{array}$$

Step4 $f(x) = (x+1)(x^2 - 2x + 2) = 0$

$x+1 = 0$ or $1x^2 - 2x + 2 = 0$

$x = -1$ or $x = \dfrac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2\cdot(1)} = \dfrac{2 \pm \sqrt{-4}}{2}$

Quadratic Formula

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow x = -1$ or $x = \dfrac{2 \pm 2i}{2} = 1 \pm i$

$$\sqrt{-4} = \underbrace{\sqrt{-1}}_{} \cdot \sqrt{4}$$

$$= i \cdot 2$$

$$= 2i$$

$\Rightarrow x = -1$ or $x = 1 + i$ or $x = 1 - i$

6. The **Number Line Test**: Given a polynomial $f(x)$

Step1. Solve the $f(x)=0$ and find all the roots.

Step2. Mark the roots on the number line and check sign (positive/negative) in each subinterval.

7. Let $f(x) = x^3 - 3x^2 + 4$. Find all the roots of $f(x)$. Sketch a complete graph and label all roots.

① possible roots (factors of 4): $1, -1, 2, -2, 4, -4$

② $f(1) = 1^3 - 3\cdot 1^2 + 4 = 1-3+4 = 2 \neq 0$

$\boxed{f(-1) = (-1)^3 - 3(-1)^2 + 4 = -1-3+4 = 0}$ ✓

$\Rightarrow x=-1$ is a root implies $(x+1)$ is a factor

③ long division

$$\begin{array}{r} x^2-4x+4 \\ x+1 \,\big)\, x^3-3x^2+0x+4 \\ -\ (x^3+x^2) \\ \hline -4x^2+0x \\ -\ (-4x^2-4x) \\ \hline 4x+4 \\ -\ (4x+4) \\ \hline 0 \end{array}$$

④ Find all roots of $f(x)$

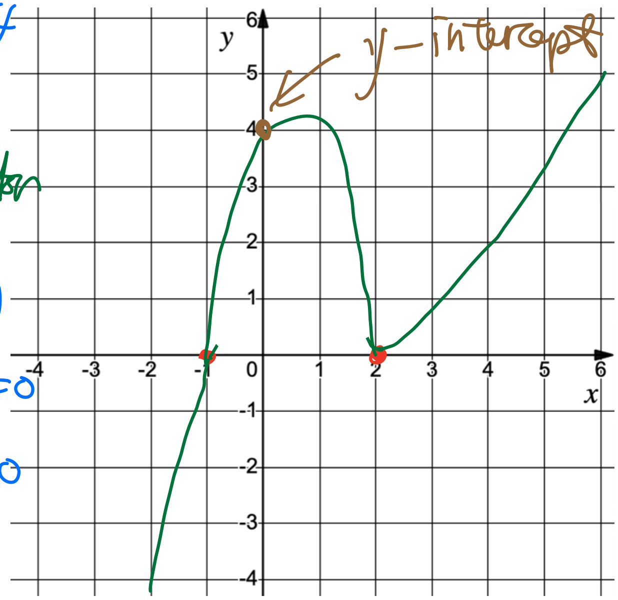
$f(x) = (x+1)(x^2-4x+4) = 0$

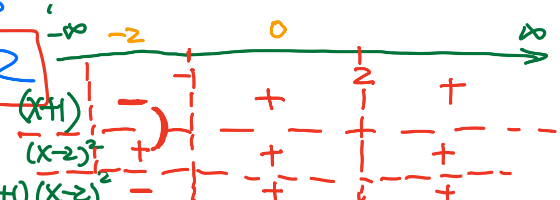
$x+1=0$ or $x^2-4x+4=0$

$x=-1$, $(x-2)(x-2)=0$

$\Rightarrow \boxed{x=-1 \text{ or } x=2, x=2}$

$f(x) = (x+1)(x-2)^2$





8. Let $f(x) = x^3 - x^2 - 9x + C$ where C is a real number. If $x = 3$ is a root of $f(x)$, find C so that $f(x)$ has this root as indicated. Then, for this choice of C, find all remaining roots of $f(x)$.

① $f(3)=0$ and Find C

$0 = f(3) = 3^3 - 3^2 - 9\cdot 3 + C$

$= 27 - 9 - 27 + C$

$= \boxed{-9+C}$

$\Rightarrow C = 9$

② long division

③ $x^3 - x^2 - 9x + 9$

$= x^2(x-1) - 9(x-1)$

$= (x-1)(x^2-9)$

$= (x-1)(x+3)(x-3)$

④ $f(x) = (x-1)(x+3)(x-3) = 0$

$\Rightarrow x-1=0$ or $x+3=0$ or $x-3=0$

$\Rightarrow x=1$ or $x=-3$ or $x=3$

9. Find a polynomial $f(x)$ that fits the given data.

$f(x)$ has degree 4. $f(x)$ has roots $0, 2, -1, -4$, and $f(1) = 20$.

$f(x)$ has roots $0, 2, -1, -4$

$\Rightarrow (x-0), (x-2), (x+1), (x+4)$ are factors of $f(x)$

$f(x) = (-2)x\cdot(x-2)(x+1)(x+4) \Longrightarrow f(x) = -2\cdot x\cdot(x-2)\cdot(x+1)(x+4)$

$f(1) = (-2)\cdot 1\cdot(1-2)(1+1)(1+4)$

$= (-2)\cdot 1\cdot(-1)\cdot 2\cdot 5 = (-2)\cdot -10 = 20$