MAT1375, Classwork7, Fall2025

Ch7. Dividing Polynomials

	degree	leading coefficient
$2x^3+5x^2+1 \Rightarrow$	3	2
$3x^4+10x^6+5x \Rightarrow$	6	10

1. Definition of **Polynomial function of degree n** in one variable:

A polynomial function of degree n in one variable is a function f of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0,$$

for some constants $a_0, a_1, \cdots, a_n$, where a_n $\neq 0$ and n is a non-negative integer. The numbers $a_0, a_1, \cdots, a_n$ are called coefficients. The **number a_n**, the coefficient of the variable to the highest power, is called the leading coefficient and $\boldsymbol{n}$ is the degree of the polynomial.

2. Rational Function.

A rational function is a fraction of two polynomials $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials, and $g(x) \neq 0$ is not the zero function. The domain of this rational function is

$D_{\frac{f}{g}} = \{x \in D_f \cap D_g,\ g(x) \neq 0\}$

3. Divide the following fraction via long division: $\frac{x^3+5x^2+4x+2}{x+3}$

$$\begin{array}{r|l}
 & x^2 + 2x - 2 \\ \hline
x+3 & x^3+5x^2+4x+2 \\
x^2(x+3) \Rightarrow & -)\ x^3+3x^2 \\ \hline
 & 2x^2+4x \\
2x(x+3) \Rightarrow & -)\ 2x^2+6x \\ \hline
 & -2x+2 \\
-2(x+3) \Rightarrow & -)\ -2x-6 \\ \hline
 & 8
\end{array}$$

$$\begin{array}{r|l}
 & 1 \quad +2 \quad -2 \\ \hline
1+3 & 1+5+4+2 \\
 & -(1+3) \\ \hline
 & 2+4 \\
 & -(2+6) \\ \hline
 & -2+2 \\
 & -(-2-6) \\ \hline
 & 8
\end{array}$$

$$\frac{x^3+5x^2+4x+2}{x+3} = x^2+2x-2 + \frac{8}{x+3}$$

or $x^3+5x^2+4x+2 = (x^2+2x-2)\cdot(x+3) + 8$

4. Dividend, Divisor, Quotient, and Remainder.

When dividing $\frac{f(x)}{g(x)}$, $f(x)$ is called the dividend and $g(x)$ is called the divisor. As a results of dividing $f(x)$ by $g(x)$ via long division with quotient $q(x)$ and remainder $r(x)$, we can write

$$\frac{f(x)}{g(x)} = \boxed{Q(x) + \frac{r(x)}{g(x)}}$$

If we multiply this equation by $g(x)$, we obtain the following alternative version:

$f(x) = Q(x)\cdot g(x) + r(x)$.

If $g(x)$ is a **factor** of $f(x)$, then we have

$f(x) = Q(x)\cdot g(x) \iff r(x) = 0$.

$x^2+2x-3 = (x-1)(x+3)$

5. Given $f(x) = \boxed{x^5 - 3x^3 + 5x^2 - 12,}$ and $g(x) = x + 3$. (a) Divide the fraction via long division: $\frac{f(x)}{g(x)}$

(b) Find $f(-3)$. (c) Is $g(x)$ a factor of $f(x)$? Why or why not?

$$
\begin{array}{r|l}
 & +x^4 - 3x^3 + 6x^2 - 13x + 39 \\
x+3 & x^5 + 0x^4 - 3x^3 + 5x^2 + 0x - 12 \\
 & -)\ x^5 + 3x^4 \\
 & -3x^4 - 3x^3 \\
 & -)\ -3x^4 - 9x^3 \\
 & 6x^3 + 5x^2 \\
 & -)\ 6x^3 + 18x^2 \\
 & -13x^2 + 0x \\
 & -)\ -13x^2 - 39x \\
 & 39x - 12 \\
 & -)\ 39x + 117 \\
 & -129
\end{array}
$$

(a) $x^5 - 3x^3 + 5x^2 - 12$
$= (x^4 - 3x^3 + 6x^2 - 13x + 39)\cdot(x+3) - 129$

(b) $f(-3) = (-3)^5 - 3(-3)^3 + 5(-3)^2 - 12$
$= -243 + 81 + 45 - 12$
$= -129$

33
-243
114
-129

(c) NO, since remainder is not 0 (which is -129)

$f(c) = q(c)(\underbrace{c-c}_{0}) + r$
$\Rightarrow f(c) = r$

6. $\boxed{\text{Remainder Theorem}}$, and Factor Theorem.

Assume $g(x) = x - c$, and the long division of $f(x)$ by $g(x)$ has remainder r, that is,

$f(x) = q(x)\cdot(x-c) + r$ (or $f(x) = q(x)\,(x-c) + r(x)$)

(1) The remainder when dividing $f(x)$ by $(x-c)$ is $f(c)$ since $f(c) = q(c)\cdot(c-c) + r = r$.

(2) $f(c) = 0 \overset{\text{if and only if}}{\Leftrightarrow}$ $g(x) = x - c$ is a factor of $f(x)$. Here c is a root of $f(x)$.

7. Given $f(x) = x^5 - 3x^3 + 5x^2 - 12$, and $g(x) = x + 2$. (a) Divide the fraction via long division: $\frac{f(x)}{g(x)}$

(b) Find $f(-2)$. (c) Is $g(x)$ a factor of $f(x)$? Why or why not?

$$
\begin{array}{r|l}
 & x^4 - 2x^3 + x^2 + 3x - 6 \\
x+2 & x^5 + 0x^4 - 3x^3 + 5x^2 + 0x - 12 \\
 & -)\ x^5 + 2x^4 \\
 & -2x^4 - 3x^3 \\
 & -)\ -2x^4 - 4x^3 \\
 & +x^3 + 5x^2 \\
 & -)\ x^3 + 2x^2 \\
 & 3x^2 + 0x \\
 & -)\ 3x^2 + 6x \\
 & -6x - 12 \\
 & -)\ -6x - 12 \\
 & 0
\end{array}
$$

$\frac{x^5}{x} = x^4$

$x^4(x+2)$

$-2x^3(x+2)$

$x^2(x+2)$

$3x(x+2)$

$-6(x+2)$

(a) $x^5 - 3x^3 + 5x^2 - 12$
$= (x^4 - 2x^3 + x^2 + 3x - 6)\cdot(x+2) + 0$

(b) $f(-2) = 0$

$f(x) = (x^4 - 2x^3 + x^2 + 3x - 6)\cdot(x+2)$
$f(-2) = (\underbrace{(-2)^4 - 2(-2)^3 + (-2)^2 + 3(-2) - 6}_{?})(\underbrace{-2+2}_{0})$

(c) Yes, since the remainder is zero.