Mat 1375 HW9

Exercise 9.3

Find the roots of the polynomial and use it to factor the polynomial completely.

✓a) $f(x) = x^3 - 7x + 6$
b) $f(x) = x^3 - x^2 - 16x - 20$
✓c) $f(x) = x^3 - 7x^2 + 17x - 20$
✓d) $f(x) = x^3 + x^2 - 5x - 2$
✓e) $f(x) = 2x^3 + x^2 - 7x - 6$
f) $f(x) = 12x^3 + 49x^2 - 2x - 24$
✓g) $f(x) = x^3 - 3x^2 + 9x + 13$
h) $f(x) = x^4 - 5x^2 + 4$

Sol. a) $f(x) = x^3 - 7x + 6$

Educational Guess of the roots: the factor of "6": $\pm1, \pm2, \pm3, \pm6$

Check:

① $x=1$, $f(1) = 1^3 - 7\cdot 1 + 6 = 0 \Rightarrow$ $x=1$ is a root and $x-1$ is a factor

Synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & +0 & -7 & +6 \\ & & +1 & +1 & -6 \\ \hline & 1 & +1 & -6 & +0 \end{array}$$

$\hookrightarrow x^2+x-6$

$\Rightarrow \boxed{f(x)} = (x-1)\cdot(x^2+x-6)$
$\boxed{= (x-1)\cdot(x+3)(x-2)}$

for $f(x)=0$, we have $x-1=0$, $x+3=0$, $x-2=0$

thus its $\boxed{\text{roots are } x=1, \ x=-3, \ x=2}$

c) $f(x) = x^3 - 7x^2 + 17x - 20$

Educational Guess of the roots: the factor of "-20": $\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$.

Check:

① $x=1$, $f(1) = 1^3 - 7\cdot1^2 + 17\cdot1 - 20 = 1 - 7 + 17 - 20 = -9 \neq 0$

② $x=-1$, $f(-1) = (-1)^3 - 7\cdot(-1)^2 + 17\cdot(-1) - 20 \neq 0$

③ $x=2$, $f(2) = 2^3 - 7\cdot2^2 + 17\cdot2 - 20 = 8 - 28 + 34 - 20 \neq 0$

④ $x=-2$, $f(-2) = (-2)^3 - 7\cdot(-2)^2 + 17(-2) - 20 \neq 0$

⑤ $x=4$, $f(4) = 4^3 - 7\cdot4^2 + 17\cdot4 - 20 = 64 - 112 + 68 - 20 = 0 \Rightarrow$

$x=4$ is a root and $x-4$ is a factor.

Synthetic division

$$\begin{array}{r|rrrr} 4 & 1 & -7 & +17 & -20 \\ & & +4 & -12 & +20 \\ \hline & 1 & -3 & +5 & 0 \end{array}$$

$\Rightarrow \boxed{f(x) = (x-4)\cdot(x^2-3x+5)}$

and, $f(x)=0$, we get the roots of f:

$(x-4)=0$ or $x^2-3x+5=0$

$$\Rightarrow \boxed{x=4 \quad \text{or} \quad x=\frac{3\pm\sqrt{9-20}}{2},\quad x=\frac{3\pm\sqrt{11}\,i}{2}}$$

(d) $f(x)=x^3+x^2-5x-2$

Educational Guess of the roots: a factor of "-2": ±1, ±2.

Check the root(s) by synthetic division (the one who has remainder $=0$)

① $x=1$.

$$\begin{array}{r|rrrr} 1 & 1 & +1 & -5 & -2 \\ & & 1 & +2 & -3 \\ \hline & 1 & +2 & -3 & -5 \end{array} \quad -5 \neq 0$$

$x=1$ is Not a root

② $x=-1$

$$\begin{array}{r|rrrr} -1 & 1 & +1 & -5 & -2 \\ & & -1 & +0 & +5 \\ \hline & 1 & +0 & -5 & 3 \end{array} \quad 3 \neq 0$$

$x=-1$ is NOT a root.

③ $x=2$

$$\begin{array}{r|rrrr} 2 & 1 & +1 & -5 & -2 \\ & & +2 & +6 & +2 \\ \hline & 1 & +3 & +1 & 0 \end{array} \quad 0 = 0$$

$x=2$ is a root and $(x-2)$ is a factor!

$$\boxed{f(x)=(x-2)\cdot(x^2+3x+1)}$$

For the root, we let $f(x)=0$ and get

$(x-2)=0$ or $x^2+3x+1=0$

$$\Rightarrow \boxed{x=2 \quad \text{or} \quad x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}}$$

(e) $f(x)=2x^3+x^2-7x-6$

Educational Guess for the root: a factor of "-6": $\pm1, \pm2, \pm3, \pm6$ $\left(\pm\frac{1}{2}, \pm\frac{3}{2}\right)$

Check the root by Synthetic division

① $x=1$,

$$\begin{array}{r|rrrr} 1 & 2 & +1 & -7 & -6 \\ & & +2 & +3 & -4 \\ \hline & 2 & 3 & -4 & -10 \end{array} \quad \neq 0$$

$x=1$ is NOT a root

② $x=-1$

$$\begin{array}{r|rrrr} -1 & 2 & +1 & -7 & -6 \\ & & -2 & +1 & +6 \\ \hline & 2 & -1 & -6 & 0 \end{array} \quad = 0$$

$x=-1$ is a root and $(x+1)$ is a factor.

$$\boxed{f(x)=(x+1)\cdot(2x^2-x-6)}$$
$$\boxed{=(x+1)\cdot(x-2)(2x+3)}$$

and the roots of $f(x)$ are

$x+1=0$, $x-2=0$, $2x+3=0$

$$\Rightarrow \boxed{x=-1,\ x=2,\ x=-\frac{3}{2}}$$

(g) $f(x) = x^3 - 3x^2 + 9x + 13$

Educational Guess of the root: a factor of "13": $\pm 1, \pm 13$

Check the root by synthetic division

① $x = 1$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & +9 & +13 \\ & & +1 & -2 & +7 \\ \hline & 1 & -2 & +7 & 20 \neq 0 \end{array}$$

$x = 1$ is NOT a root

② $x = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & +9 & +13 \\ & & -1 & +4 & -13 \\ \hline & 1 & -4 & +13 & 0 = 0 \end{array}$$

$x = -1$ is a root and $(x+1)$ is a factor

$\boxed{f(x)} = x^3 - 3x^2 + 9x + 13$ $\boxed{= (x+1)\cdot(x^2 - 4x + 13)}$ and $\boxed{\text{its roots}}$ are

$x + 1 = 0$ and $x^2 - 4x + 13 = 0$

$\Rightarrow \boxed{x = -1}$, $x = \dfrac{4 \pm \sqrt{16 - 52}}{2}$

, $x = \dfrac{4 \pm 6i}{2}$

$\boxed{x = 2 \pm 3i}$

Exercise 9.4

Find the exact roots of the polynomial; write the roots in simplest radical form, if necessary. Sketch a graph of the polynomial with all roots clearly marked.

✓a) $f(x) = x^3 - 2x^2 - 5x + 6$

✓b) $f(x) = x^3 + 5x^2 + 3x - 4$

c) $f(x) = -x^3 + 5x^2 + 7x - 35$

d) $f(x) = x^3 + 7x^2 + 13x + 7$

✓e) $f(x) = 2x^3 - 8x^2 - 18x - 36$

f) $f(x) = x^4 - 4x^2 + 3$

Sol. a) $f(x) = x^3 - 2x^2 - 5x + 6$

Educational Guess of roots: a factor of "6": $\pm 1, \pm 2, \pm 3, \pm 6$

Check roots by synthetic division:

① $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & +6 \\ & & +1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 = 0 \end{array}$$

$x = 1$ is a roots and $(x-1)$ is a factor

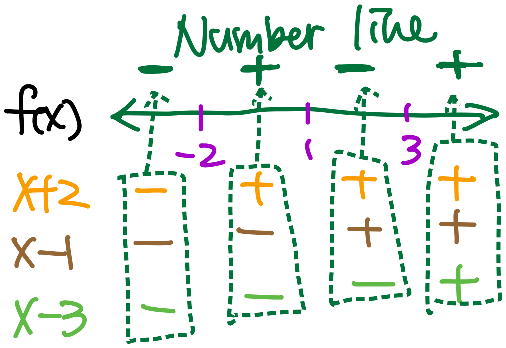
$f(x) = x^3 - 2x^2 - 5x + 6 = (x-1)\cdot(x^2 - x - 6) = (x-1)\cdot(x+2)(x-3)$

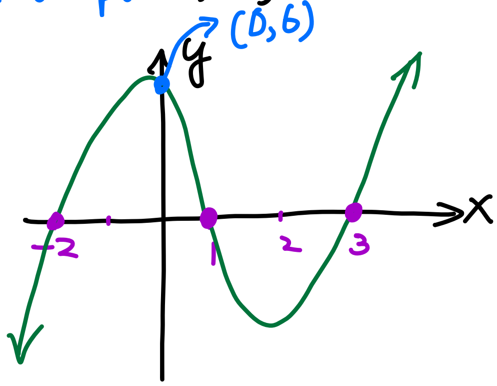
$\Rightarrow$ the roots of $f(x)$ is $x - 1 = 0$, $x + 2 = 0$, $x - 3 = 0$

$\Rightarrow x = 1$, $x = -2$, $x = 3$.

Graph:

y-intercept: $f(0)=0^3-2\cdot 0^2-5\cdot 0+6=6 \Rightarrow (0,6)$





(b) $f(x)=x^3+5x^2+3x-4$

Educational Guess of a root: a factor of "−4": $\pm 1, \pm 2, \pm 4$.

Check roots by synthetic division:

① $x=1$.

$$\begin{array}{r|rrrr} 1 & 1 & +5 & +3 & -4 \\ & & +1 & +6 & +9 \\ \hline & 1 & +6 & +9 & 5 \end{array} \neq 0$$

$x=1$ is NOT a root

② $x=-1$

$$\begin{array}{r|rrrr} -1 & 1 & +5 & +3 & -4 \\ & & -1 & -4 & +1 \\ \hline & 1 & +4 & -1 & -3 \end{array} \neq 0$$

$x=-1$ is NOT a root

③ $x=2$

$$\begin{array}{r|rrrr} 2 & 1 & +5 & +3 & -4 \\ & & +2 & +7 & +20 \\ \hline & 1 & +7 & +10 & 24 \end{array} \neq 0$$

$x=2$ is NOT a root

④ $x=-2$

$$\begin{array}{r|rrrr} -2 & 1 & +5 & +3 & -4 \\ & & -2 & -6 & +6 \\ \hline & 1 & +3 & -3 & 2 \end{array} \neq 0$$

$x=-2$ is NOT a root

⑤ $x=4$

$$\begin{array}{r|rrrr} 4 & 1 & +5 & +3 & -4 \\ & & +4 & +36 & +156 \\ \hline & 1 & +9 & +39 & 152 \end{array} \neq 0$$

$x=4$ is NOT a root

⑥ $x=-4$

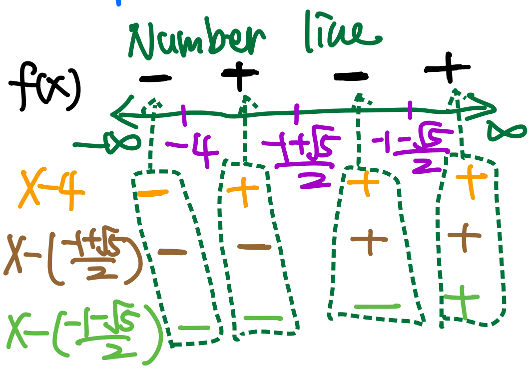
$$\begin{array}{r|rrrr} -4 & 1 & +5 & +3 & -4 \\ & & -4 & -4 & +4 \\ \hline & 1 & +1 & -1 & 0 \end{array} = 0$$

$x=-4$ is a root and $x+4$ is a factor.

$f(x)=(x+4)\cdot(x^2+x-1)$ and its roots are $x+4=0$ and $x^2+x-1=0$

$\Rightarrow x=-4$ and $x=\dfrac{-1\pm\sqrt{1+4}}{2}=\dfrac{-1\pm\sqrt{5}}{2}$

Graph:

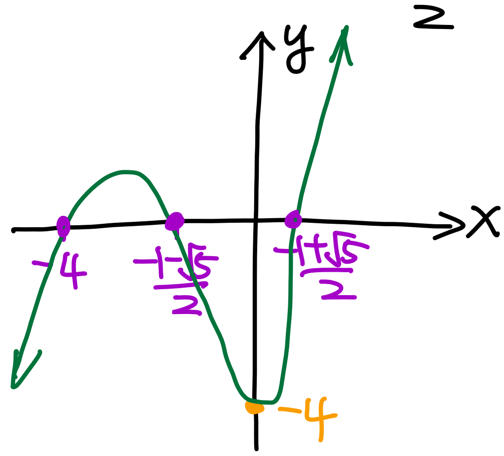


y-intercept

$f(0)=-4$

$\Downarrow$

$(0,-4)$



(e) $f(x) = 2x^3 - 8x^2 - 18x - 36$

Educational Guess of a root: a factor of "-36": $\pm1, \pm2, \pm3, \pm4, \pm9, \pm12, \pm18, \pm36$.

$\left(\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm6\right)$

Check root(s) by synthetic division.

① $x=1$.

$$\begin{array}{r|rrrr} 1 & 2 & -8 & -18 & -36 \\ & & +2 & -6 & -24 \\ \hline & 2 & -6 & -24 & -60 \end{array} \neq 0$$

NOT a root

② $x=-1$

$$\begin{array}{r|rrrr} -1 & 2 & -8 & -18 & -36 \\ & & -2 & +10 & +8 \\ \hline & 2 & -10 & -8 & -28 \end{array} \neq 0$$

NOT a root

③ $x=2$

$$\begin{array}{r|rrrr} 2 & 2 & -8 & -18 & -36 \\ & & +4 & -8 & -52 \\ \hline & 2 & -4 & -26 & -88 \end{array} \neq 0$$

NOT a root

④ $x=-2$

$$\begin{array}{r|rrrr} -2 & 2 & -8 & -18 & -36 \\ & & -4 & +24 & -12 \\ \hline & 2 & -12 & +6 & -48 \end{array} \neq 0$$

NOT a root

⑤ $x=3$

$$\begin{array}{r|rrrr} 3 & 2 & -8 & -18 & -36 \\ & & +6 & -6 & -72 \\ \hline & 2 & -2 & -24 & -108 \end{array} \neq 0$$

NOT a root

⑥ $x=-3$

$$\begin{array}{r|rrrr} -3 & 2 & -8 & -18 & -36 \\ & & -6 & +42 & -72 \\ \hline & 2 & -14 & +24 & -108 \end{array} \neq 0$$

NOT a root.

⑦ $x=6$

$$\begin{array}{r|rrrr} 6 & 2 & -8 & -18 & -36 \\ & & +12 & +24 & +36 \\ \hline & 2 & +4 & +6 & 0 \end{array} = 0$$

$x=6$ is a root and $(x-6)$ is a factor

$$f(x) = 2x^3 - 8x^2 - 18x - 36 = (x-6)\cdot(2x^2+4x+6)$$
$$= 2\cdot(x-6)(x^2+2x+3)$$

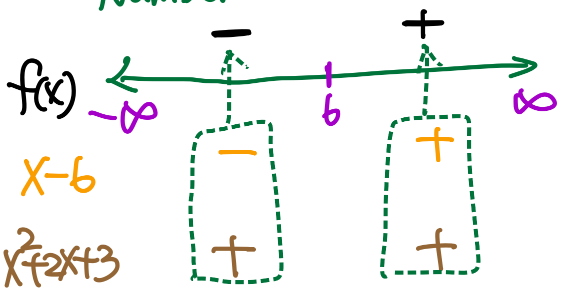
and its roots are

$x-6=0$ and $x^2+2x+3=0 \Rightarrow \boxed{x=6}$ and $x = \frac{-2\pm\sqrt{4-12}}{2} = \frac{-2\pm2\sqrt{2}i}{2}$

$$\boxed{x = -1\pm\sqrt{2}i}$$

Graph

Number line

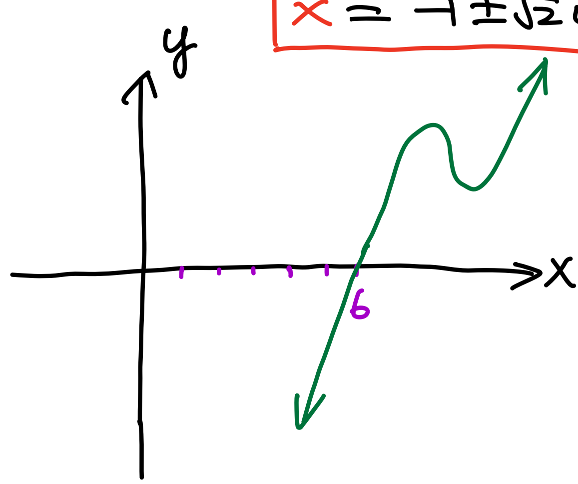


y-intercept

$f(0) = -36$

$\Downarrow$

$(0, -36)$



Exercise 9.5

Find a real number C so that the polynomial has a root as indicated. Then, for this choice of C, find all remaining roots of the polynomial.

a) $f(x) = x^3 + 6x^2 + 5x + C$ has root at $x = 1$
b) $f(x) = x^3 - 4x^2 - 2x + C$ has root at $x = -2$
c) $f(x) = x^3 - x^2 - 9x + C$ has root at $x = 3$
d) $f(x) = x^3 + 8x^2 + 5x + C$ has root at $x = -1$

Sol. a) f(x) has a root at $x=1 \Rightarrow f(1)=0$.

$$0=f(1)=(1)^3+6(1)^2+5\cdot 1+C=1+6+5+C=12+C$$

$$\Rightarrow 0=12+C \Rightarrow \boxed{C=-12}$$

$$\begin{array}{r|rrrr} 1 & 1 & +6 & +5 & -12 \\ & & +1 & +7 & +12 \\ \hline & 1 & +7 & +12 & \boxed{0} \end{array}$$

$$f(x)=x^3+6x^2+5x-12=(x-1)(x^2+7x+12)$$
$$=(x-1)(x+3)(x+4)$$

and its roots are $x-1=0,\ x+3=0,\ x+4=0$
$$\Rightarrow x=1,\ x=-3,\ x=-4$$

b) f(x) has a root at $x=-2 \Rightarrow f(-2)=0$

$$0=f(-2)=(-2)^3-4\cdot(-2)^2-2\cdot(-2)+C=-8-16+4+C=-20+C$$

$$\Rightarrow 0=-20+C \Rightarrow \boxed{C=20}$$

$$\begin{array}{r|rrrr} -2 & 1 & -4 & -2 & +20 \\ & & -2 & +12 & -20 \\ \hline & 1 & -6 & +10 & \boxed{0} \end{array}$$

$$f(x)=x^3-4x^2-2x+20=(x+2)\cdot(x^2-6x+10)$$

and its roots are $x+2=0$ and $x^2-6x+10=0$
$$\Rightarrow \boxed{x=-2} \text{ and } x=\frac{6\pm\sqrt{36-40}}{2}$$
$$=\frac{6\pm 2i}{2}=\boxed{3\pm i}$$

d) f(x) has a root $x=-1 \Rightarrow f(-1)=0$

$$0=f(-1)=(-1)^3+8(-1)^2+5(-1)+C=-1+8-5+C$$

$$\Rightarrow 0=2+C \Rightarrow \boxed{C=-2}$$

$$\begin{array}{r|rrrr} -1 & 1 & +8 & +5 & -2 \\ & & -1 & -7 & +2 \\ \hline & 1 & +7 & -2 & \boxed{0} \end{array}$$

$$f(x)=x^3+8x^2+5x-2=(x+1)\cdot(x^2+7x-2)$$

and its roots are $x+1=0$ and $x^2+7x-2=0$

$\Rightarrow \boxed{x=-1}$ and $x=\frac{-7\pm\sqrt{49+8}}{2}=\boxed{\frac{-7\pm\sqrt{57}}{2}}$

Exercise 9.6

Find a polynomial f that fits the given data.

a) f has degree 3. The roots of f are precisely 2, 3, 4. The leading coefficient of f is 2.

b) f has degree 4. The roots of f are precisely -1, 2, 0, -3. The leading coefficient of f is -1.

c) f has degree 3. f has roots -2, -1, 2, and $f(0)=10$.

a) The roots of f are $2, 3, 4$ $\Rightarrow$ factors: $(x-2)$, $(x-3)$, $(x-4)$

$\deg(f)=3$, $\Rightarrow$ total 3 roots, no repeat root.

leading coefficient is 2 $\Rightarrow \boxed{f(x)=2\cdot(x-2)(x-3)\cdot(x-4)}$

b) The roots of f are $-1, 2, 0, -3$ $\Rightarrow$ factors: $(x+1)$, $(x-2)$, x, $(x+3)$

$\deg(f)=4$ $\Rightarrow$ total 4 roots, no repeat one

leading coeff. $=-1$ $\Rightarrow \boxed{f(x)=-1\cdot(x+1)(x-2)\cdot x\cdot(x+3)}$

c) The roots of f are $-2, -1, 2$ $\Rightarrow$ factors: $(x+2)$, $(x+1)$, $(x-2)$

$\deg(f)=3$ $\Rightarrow$ total 3 roots, no repeat one

$f(0)=10$ $\Rightarrow$ let $f(x)=C\cdot(x+2)(x+1)(x-2)$

$\Rightarrow 10=f(0)=C\cdot(0+2)(0+1)(0-2)=C\cdot(2)\cdot(1)\cdot(-2)=-4C$

$\Rightarrow -4C=10 \Rightarrow C=-\frac{5}{2}$ and $\boxed{f(x)=-\frac{5}{2}(x+2)(x+1)(x-2)}$

g) f has degree 4. The coefficients of f are all real. f has roots $5+i$ and $5-i$ of multiplicity 1, the root 3 of multiplicity 2, and $f(5)=7$.

h) f has degree 4. The coefficients of f are all real. f has roots i and $3+2i$.

i) f has degree 6. f has complex coefficients. f has roots $1+i$, $2+i$, $4-3i$ of multiplicity 1 and the root -2 of multiplicity 3.

g) f has roots $5+i$, $5-i$ (multi = 1), $3, 3$ (multi = 2)

$\Rightarrow f$ has factor $(x-5-i)$, $(x-5+i)$, $(x-3)$, $(x-3)$

$f(5)=7 \Rightarrow$ let $f(x) = C\cdot(x-5-i)\cdot(x-5+i)(x-3)^2$

$= C\cdot((x-5)^2+1)(x-3)^2 = C\cdot(x^2-10x+26)(x-3)^2$

$7 = f(5) = C\cdot((5-5)^2+1)\cdot(5-3)^2 = C\cdot(1)\cdot(2)^2 = 4C$

$\Rightarrow C = \frac{7}{4}$ and $\boxed{f(x) = \frac{7}{4}(x^2-10x+26)(x-3)^2}$

h) f (a polynomial with real coefficients)

f has root $i \Rightarrow$ has root "$-i$" as well $\Rightarrow$ factors $(x+i)$, $(x-i)$

f has root $3+2i \Rightarrow$ has root "$3-2i$" as well $\Rightarrow$ factor $(x-3-2i)$, $(x-3+2i)$

$f(x) = (x+i)(x-i)(x-3-2i)(x-3+2i)$

$= (x^2+1)((x-3)^2+4) = \boxed{(x^2+1)(x^2-6x+13)}$

i) f has roots $1+i$, $2+i$, $4-3i$, $-2, -2, -2$ (multi = 3)

f has factors $(x-1-i)(x-2-i)$, $(x-4+3i)$, $(x+2)$, $(x+2)$, $(x+2)$

$\Rightarrow \boxed{f(x) = (x-1-i)(x-2-i)(x-4+3i)\cdot(x+2)^3}$