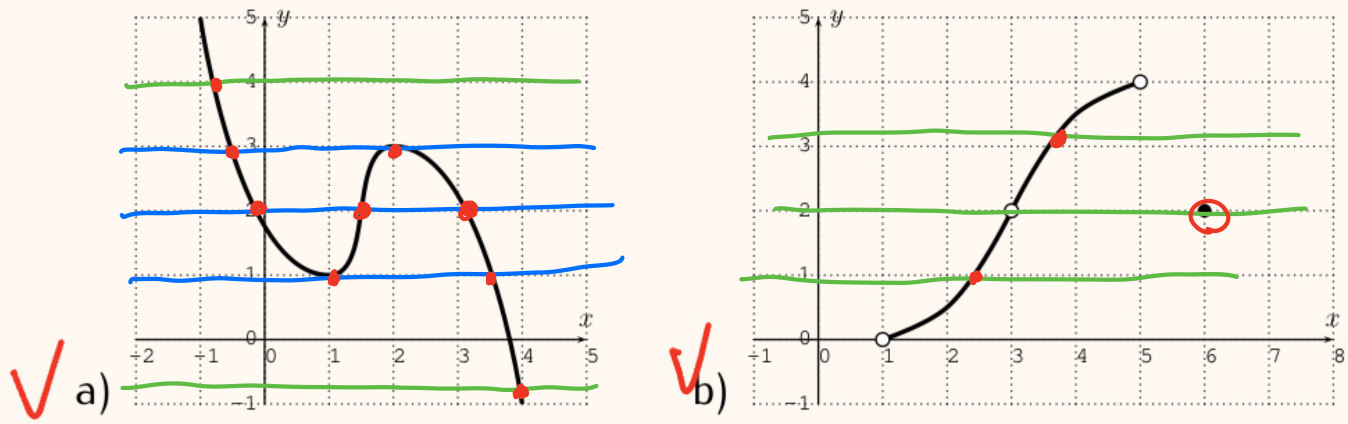
Mat 1375 HW6

Exercise 6.1

Use the horizontal line test to determine whether the function is one-to-one.

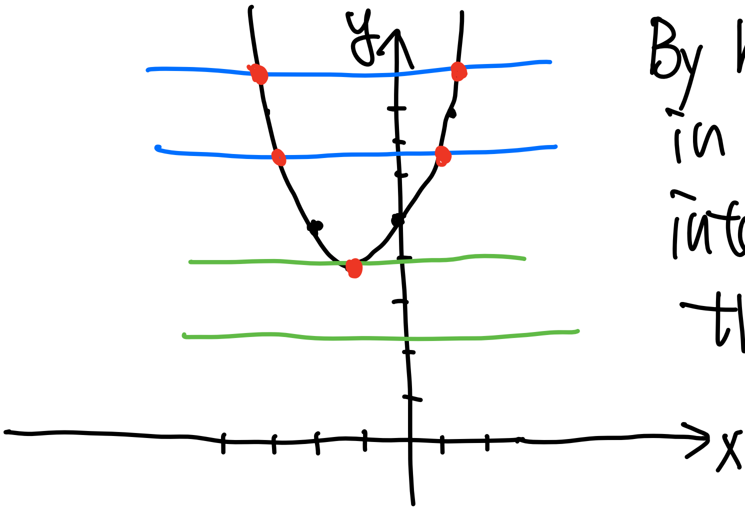


Sol (a) By horizontal line test, for those horizontal lines in blue, there are more than one intersection point with the given graph, thus, it is NOT one-to-one

(b) By horizontal line test, for each horizontal line there is at most one intersection point with the given graph, thus, it is 1-to-1.

c) $f(x) = x^2 + 2x + 5$ $\Rightarrow$ $f(x) = x^2+2x+1+4 = (x+1)^2+4$

which is $f(x)=x^2$ shifting left 1 unit and shifting up 4 units



By horizontal line test, those lines in blue have more than one intersection point with $f(x)$, thus, $f(x)$ is NOT one-to-one

Exercise 6.2

Find the inverse of the function f and check your solution.

✓ a) $f(x) = 4x + 9$	✓ b) $f(x) = -8x - 3$
✓ c) $f(x) = \sqrt{x + 8}$	✓ d) $f(x) = \sqrt{3x + 7}$
✓ e) $f(x) = 6 \cdot \sqrt{-x - 2}$	✓ f) $f(x) = x^3$

Sol. To find the inverse function. we have 4-step process.

Step 1 replace "$f(x)$" by "y"

Step 2 switch x and y

Step 3 Solve for y

Step 4 replace "y" by "$f^{-1}(x)$"

Check:

$f(f^{-1}(x)) = x$

and

$f^{-1}(f(x)) = x$

(a) $f(x) = 4x+9$

Step 1 $y = 4x+9$

Step 2 $x = 4y+9$

Step 3 $4y = x - 9 \Rightarrow y = \frac{x-9}{4}$

Step 4 $\boxed{f^{-1}(x) = \frac{x-9}{4}}$

Check.

$f(f^{-1}(x)) = 4\cdot(f^{-1}(x)) + 9$

$= 4\cdot\left(\frac{x-9}{4}\right) + 9$

$= (x-9) + 9$

$= x - 9 + 9 = x$

and

$f^{-1}(f(x)) = \frac{(f(x)) - 9}{4}$

$\Rightarrow f^{-1}(x) = \dfrac{x-9}{4}$ is the inverse of $f(x)$.

$= \dfrac{(4x+9)-9}{4}$

$= \dfrac{4x+9-9}{4} = \dfrac{4x}{4} = x$

(b) $f(x) = -8x - 3$

Step1 $\quad y = -8x - 3$

Step2 $\quad x = -8y - 3$

Step3 $\quad -8y = x + 3$

$\Rightarrow y = \dfrac{x+3}{-8}$

Step4 $\quad \boxed{f^{-1}(x) = \dfrac{x+3}{-8}}$

check:

$\boxed{f(f^{-1}(x))} = -8(f^{-1}(x)) - 3$

$= -8\left(\dfrac{x+3}{-8}\right) - 3$

$= (x+3) - 3 = \boxed{x}$

and

$\boxed{f^{-1}(f(x))} = \dfrac{(f(x)) + 3}{-8} = \dfrac{(-8x-3)+3}{-8}$

$= \dfrac{-8x-3+3}{-8} = \dfrac{-8x}{-8} = \boxed{x}$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then

$f^{-1}(x) = \dfrac{x+3}{-8}$ is the inverse of f.

(c) $f(x) = \sqrt{x+8}$ $\quad (x \geq -8)$

Step1 $\quad y = \sqrt{x+8}$ $\quad (x \geq -8)$

Step2 $\quad x = \sqrt{y+8}$ $\quad (y \geq -8)$ and $x \geq 0$

Step3 $\quad (x)^2 = \left(\sqrt{y+8}\right)^2$ $\quad (y \geq -8)$ and $x \geq 0$

$\Rightarrow x^2 = y + 8$

$\Rightarrow y = x^2 - 8$

check:

$f(f^{-1}(x)) = \sqrt{(f^{-1}(x)) + 8}$

$= \sqrt{(x^2 - 8) + 8}$

$= \sqrt{x^2 - 8 + 8}$

$= \sqrt{x^2} = x$ $\quad x \geq 0$

and

$f^{-1}(f(x)) = (f(x))^2 - 8$

Step4 $\boxed{f^{-1}(x) = x^2 - 8 \quad \text{and } x \geq 0}$ $= (\sqrt{x+8})^2 - 8$

$= x + 8 - 8 = x$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then

$f^{-1}(x) = x^2 - 8$ is the inverse of f.

and $x \geq 0$

(d) $f(x) = \sqrt{3x+7}$ $\left(x \geq -\frac{7}{3} \right)$

Step1 $y = \sqrt{3x+7}$

Step2 $x = \sqrt{3y+7}$ $\left(y \geq -\frac{7}{3} \right)$

and $x \geq 0$

Step3 $(x)^2 = \left(\sqrt{3y+7}\right)^2$

$x^2 = 3y + 7 \Rightarrow 3y = x^2 - 7$

$\Rightarrow y = \frac{x^2-7}{3}$

Step4 $\boxed{f^{-1}(x) = \frac{x^2-7}{3} \quad (x \geq 0)}$

Check:

$f(f^{-1}(x)) = \sqrt{3 \cdot f^{-1}(x) + 7}$

$= \sqrt{3 \cdot \left(\frac{x^2-7}{3}\right) + 7}$

$= \sqrt{(x^2 - 7) + 7}$

$= \sqrt{x^2} = x \quad (x \geq 0)$

and

$f^{-1}(f(x)) = \frac{(f(x))^2 - 7}{3}$

$= \frac{(\sqrt{3x+7})^2 - 7}{3}$

$= \frac{3x + 7 - 7}{3} = \frac{3x}{3} = x$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then

$f^{-1}(x) = \frac{x^2-7}{3}$ is the inverse of f.

(e) $f(x) = 6\cdot\sqrt{-x-2}$ ($x \le -2$)

$\underline{\text{step1}}$ $y = 6\cdot\sqrt{-x-2}$

$\underline{\text{step2}}$ $x = 6\cdot\sqrt{-y-2}$ ($y \le -2$)
and $x \ge 0$

$\underline{\text{step3}}$ $(x)^2 = \left(6\cdot\sqrt{-y-2}\right)^2$

$$x^2 = 36\cdot(-y-2)$$

$$\Rightarrow x^2 = -36y - 72$$

$$\Rightarrow -36y = x^2 + 72$$

$$\Rightarrow y = \frac{x^2+72}{-36}$$

$\underline{\text{step4}}$ $\boxed{f^{-1}(x) = \frac{x^2+72}{-36}}$

$\underline{\text{check}}$

$$f(f^{-1}(x)) = 6\cdot\sqrt{-f^{-1}(x)-2} = 6\cdot\sqrt{-\left(\frac{x^2+72}{-36}\right)-2}$$

$$= 6\cdot\sqrt{\frac{x^2+72}{36}-2} = 6\sqrt{\frac{x^2}{36}+\frac{72}{36}-2}$$

$$= 6\cdot\sqrt{\frac{x^2}{36}} = \cancel{6}\,\frac{x}{\cancel{6}} = x$$

and

$$f^{-1}(f(x)) = \frac{\left(6\cdot\sqrt{-x-2}\right)^2+72}{-36}$$

$$= \frac{36\cdot(-x-2)+72}{-36} = \frac{-36x-72+72}{-36} = \frac{-36x}{-36} = x$$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then

$f^{-1}(x) = \frac{x^2+72}{-36}$ is the inverse of f

f) $f(x) = x^3$

Check

Step 1 $y = x^3$

$f(f^{-1}(x)) = (\sqrt[3]{x})^3 = x$ and

Step 2 $x = y^3$

$f^{-1}(f(x)) = \sqrt[3]{x^3} = x$

Step 3 $\sqrt[3]{y^3} = \sqrt[3]{x}$

$\Rightarrow y = \sqrt[3]{x}$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then $f^{-1} = \sqrt[3]{x}$ is the inverse of f

Step 4 $f^{-1}(x) = \sqrt[3]{x}$

✓ l) $f(x) = \frac{-5}{4-x}$ ✓ m) $f(x) = \frac{x}{x+2}$ ✓ n) $f(x) = \frac{3x}{x-6}$

✓ o) $f(x) = \frac{x+2}{x+3}$ ✓ p) $f(x) = \frac{7-x}{x-5}$ q) f given by the

l) $f(x) = \frac{-5}{4-x}$ $(x \neq 4)$

Step 1 $y = \frac{-5}{4-x}$

Step 2 $x = \frac{-5}{4-y}$ $(y \neq 4)$

Step 3 $x(4-y) = -5$

$4x - xy = -5$

$xy = 4x + 5$

$y = \frac{4x+5}{x}$ $(x \neq 0)$

Step 4 $f^{-1}(x) = \frac{4x+5}{x}$

Check

$f(f^{-1}(x)) = \frac{-5}{4 - f^{-1}(x)}$

$= \frac{-5}{4 - \frac{4x+5}{x}} = \frac{-5}{\frac{4x-4x-5}{x}}$

$= \frac{-5}{\frac{-5}{x}} = -5 \cdot \frac{x}{-5} = x$

and

$f^{-1}(f(x)) = \frac{4\left(\frac{-5}{4-x}\right)+5}{\left(\frac{-5}{4-x}\right)}$

$= \frac{\frac{-20+5(4-x)}{(4-x)}}{\frac{-5}{4-x}}$

$$= \frac{\frac{-20+20-5x}{4-x}}{\frac{-5}{4-x}} = \frac{\frac{-5x}{4-x}}{\frac{-5}{4-x}} = x$$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then $f^{-1} = \frac{4x+5}{x}$ is the inverse of f.

m) $f(x) = \frac{x}{x+2}$ $(x \neq -2)$

Step1 $y = \frac{x}{x+2}$

Step2 $\frac{x}{1} = \frac{y}{y+2}$ $(y \neq -2)$

Step3 $x(y+2) = y$

$xy + 2x = y$

$\Rightarrow xy - y = -2x$

$\Rightarrow y(x-1) = -2x$

$\Rightarrow y = \frac{-2x}{x-1}$

Step4 $f^{-1}(x) = \frac{-2x}{x-1}$

Check

$$f(f^{-1}(x)) = \frac{f^{-1}(x)}{f^{-1}(x)+2} = \frac{\frac{-2x}{x-1}}{\frac{-2x}{x-1}+2} = \frac{\frac{-2x}{x-1}}{\frac{-2x+2(x-1)}{x-1}} = \frac{\frac{-2x}{x-1}}{\frac{-2x+2x-2}{x-1}} = \frac{\frac{-2x}{x-1}}{\frac{-2}{x-1}} = \frac{-2x}{x-1} \cdot \frac{x-1}{-2} = x$$

and

$$f^{-1}(f(x)) = \frac{-2f(x)}{f(x)-1} = \frac{-2\frac{x}{x+2}}{\frac{x}{x+2}-1} = \frac{\frac{-2x}{x+2}}{\frac{x-(x+2)}{x+2}} = \frac{\frac{-2x}{x+2}}{\frac{-2}{x+2}} = \frac{-2x}{x+2} \times \frac{x+2}{-2} = x.$$

Since $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$, then

$f^{-1}=\frac{-2x}{x-1}$ is the inverse of f.

h) $f(x)=\frac{3x}{x-6}$ $(x\neq 6)$

Step 1 $y=\frac{3x}{x-6}$

Step 2 $\frac{x}{1}=\frac{3y}{y-6}$ $(y\neq 6)$

Step 3 $x(y-6)=3y$

$xy-6x=3y$

$xy-3y=6x$

$y(x-3)=6x$

$y=\frac{6x}{x-3}$ $(x\neq 3)$

Step 4 $f^{-1}(x)=\frac{6x}{x-3}$

Check:

$$f(f^{-1}(x))=\frac{3f^{-1}(x)}{f^{-1}(x)-6}=\frac{3\frac{6x}{x-3}}{\frac{6x}{x-3}-6}=\frac{\frac{18x}{x-3}}{\frac{6x-6(x-3)}{x-3}}=\frac{\frac{18x}{x-3}}{\frac{18}{x-3}}=\frac{18x}{x-3}\times\frac{x-3}{18}=x$$

and

$$f^{-1}(f(x))=\frac{6f(x)}{f(x)-3}=\frac{6\frac{3x}{x-6}}{\frac{3x}{x-6}-3}$$

$$=\frac{\frac{18x}{x-6}}{\frac{3x-3(x-6)}{x-6}}=\frac{\frac{18x}{x-6}}{\frac{18}{x-6}}=\frac{18x}{x-6}\cdot\frac{x-6}{18}=x$$

Since $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$, then

$f^{-1}=\frac{6x}{x-3}$ is the inverse of f.

o) $f(x) = \frac{x+2}{x+3}$ $(x \neq -3)$

Step1 $y = \frac{x+2}{x+3}$

Step2 $\frac{x}{1} = \frac{y+2}{y+3}$ $(y \neq -3)$

Step3 $x(y+3) = y+2$

$\Rightarrow xy + 3x = y+2$

$\Rightarrow xy - y = 2-3x$

$\Rightarrow y(x-1) = 2-3x$

$\Rightarrow y = \frac{2-3x}{x-1}$

Step4 $f^{-1}(x) = \frac{2-3x}{x-1}$

Check

$$f(f^{-1}(x)) = \frac{f^{-1}(x)+2}{f^{-1}(x)+3}$$

$$= \frac{\frac{2-3x}{x-1}+2}{\frac{2-3x}{x-1}+3}$$

$$= \frac{\frac{2-3x+2(x-1)}{\cancel{x-1}}}{\frac{2-3x+3(x-1)}{\cancel{x-1}}}$$

$$= \frac{2-3x+2x-2}{2-3x+3x-3} = \frac{-x}{-1} = x$$

and

$$f^{-1}(f(x)) = \frac{2-3f(x)}{f(x)-1}$$

$$= \frac{2-3\frac{x+2}{x+3}}{\frac{x+2}{x+3}-1}$$

$$= \frac{\frac{2(x+3)-3(x+2)}{\cancel{x+3}}}{\frac{x+2-(x+3)}{\cancel{x+3}}} = \frac{2x+6-3x-6}{x+2-x-3} = \frac{-x}{-1} = x$$

Since $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then

$f^{-1} = \frac{2-3x}{x-1}$ is the inverse of f.

Exercise 6.3

Restrict the domain of the function f in such a way that f becomes a one-to-one function. Find the inverse of f with the restricted domain.

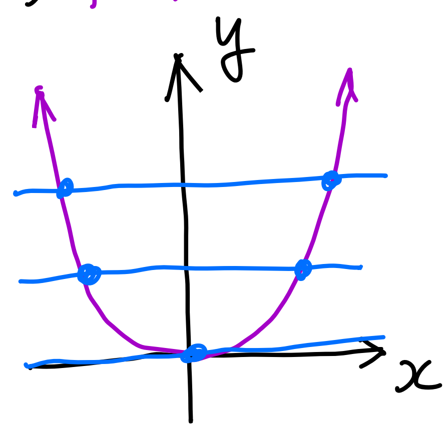
✓ a) $f(x) = x^2$ ✓ b) $f(x) = (x+5)^2 + 1$

✓ c) $f(x) = |x|$ d) $f(x) = |x-4| - 2$

Sol How to find the proper restriction?

$\Rightarrow$ Graphing f(x) then testing it by horizontal line test.

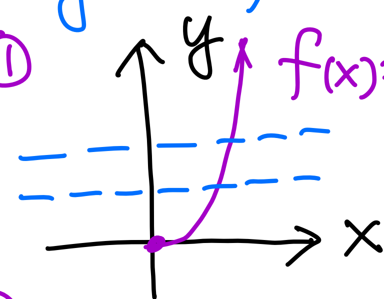
a) $f(x) = x^2$



two options

$x \geq 0$

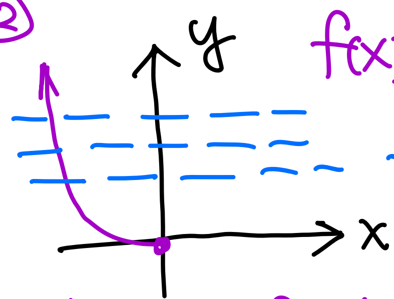
① $f(x)=x^2$, but $x \geq 0$



$\Rightarrow$ pass the test

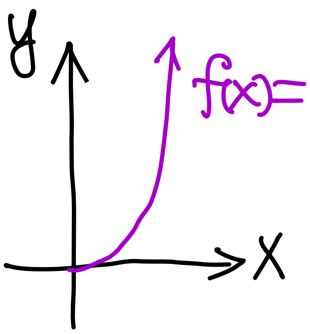
$x \leq 0$

② $f(x)=x^2$, but $x \leq 0$



$\Rightarrow$ pass the test.

I will use ① as an example to find its inverse.



$f(x)=x^2$, $x \geq 0$

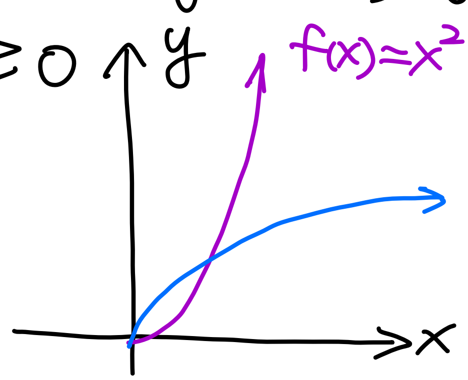
step 1 $y = x^2$, $x \geq 0$

step 2 $x = y^2$, $y \geq 0$

step 3 $\sqrt{y^2} = \sqrt{x}$, $y \geq 0$, $x \geq 0$

$\Rightarrow y = \sqrt{x}$, $y \geq 0$, $x \geq 0$

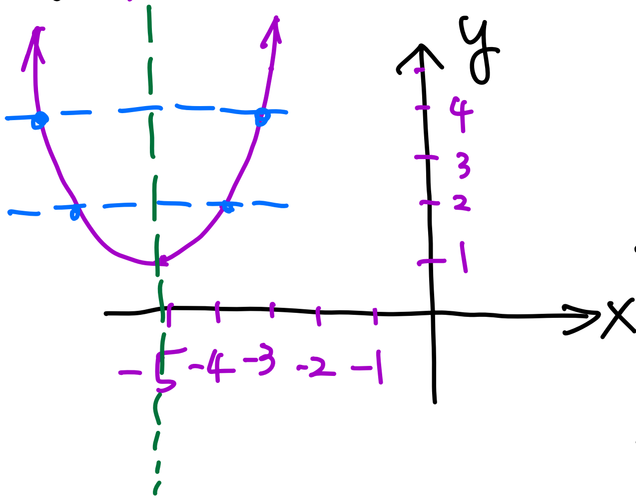
step 4 $f^{-1}(x) = \sqrt{x}$, $x \geq 0$



$f(x)=x^2$, $x \geq 0$

$f^{-1}(x) = \sqrt{x}$, $x \geq 0$

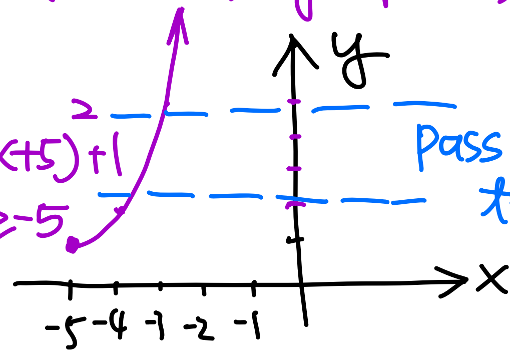
b) $f(x)=(x+5)^2+1$ (which is $f(x)=x^2$ shifting 5 to the left and shifting up 1)



two options

① $x \geq -5$

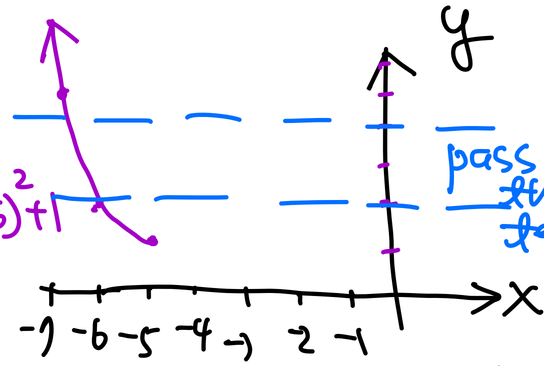
$f(x)=(x+5)^2+1$
but $x \geq -5$



Pass the test

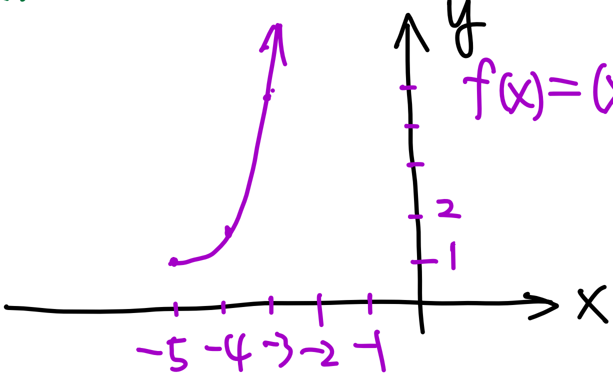
② $x \leq -5$

$f(x)=(x+5)^2+1$
$x \leq -5$



pass the test

I will use ① as an example to find the inverse function



$f(x)=(x+5)^2+1$, but $x \geq -5$

<u>step1</u> $y=(x+5)^2+1$, $x \geq -5$

<u>step2</u> $x=(y+5)^2+1$, $y \geq -5$

<u>step3</u> $x=y^2+10y+25+1$, $y \geq -5$

$\Rightarrow y^2+10y+26-x=0$, $y \geq -5$

$y=\dfrac{-10 \pm \sqrt{100-4 \cdot (26-x)}}{2}$, $y \geq -5$

$=\dfrac{-10 \pm \sqrt{100-104+4x}}{2}$, $y \geq -5$

$=\dfrac{-10 \pm \sqrt{4x-4}}{2}=\dfrac{-10 \pm 2\sqrt{x-1}}{2}=-5 \pm \sqrt{x-1}$, $y \geq -5$

Since $2\sqrt{x-1}$ is always positive, and it will make

$y = -5 - \sqrt{x-1}$ to be less than -5,

and $y = -5 + \sqrt{x-1}$ to be more than -5,

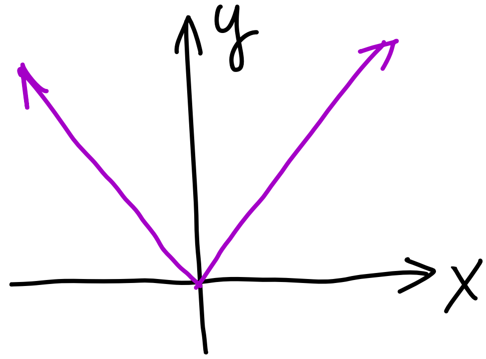
Then $y = -5 + \sqrt{x-1}$ is the one we want.

Step4 $\boxed{f^{-1}(x) = -5 + \sqrt{x-1}, \quad x \geq 1}$

The inverse of $f(x) = (x+5)^2 + 1$, $x \geq -5$

is $f^{-1}(x) = -5 + 2\sqrt{x+1}$, $x \geq 1$.

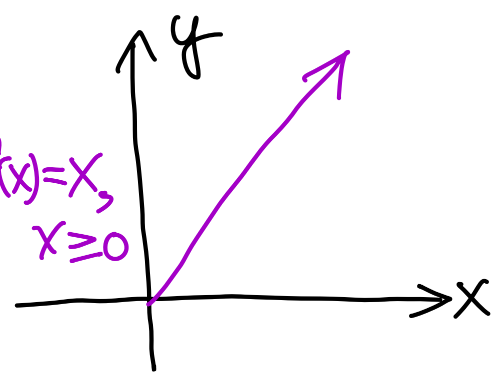
c) $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$



two options

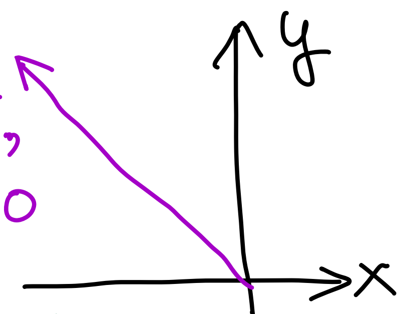
① $x \geq 0$

$f(x) = x$, $x \geq 0$

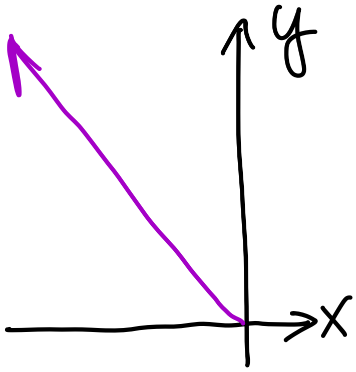


② $x \leq 0$

$f(x) = -x$, $x \leq 0$



I will use ② as an example to find the inverse.



$f(x) = -x$, $x \leq 0$

Step1 $y = -x$, $x \leq 0$

Step2 $x = -y$, $y \leq 0$

Step3 $y = -x$, $y \leq 0 \Rightarrow y = -x$, $x \geq 0$

↳ if $y \leq 0$ and $y = -x$, it implies step4 $f^{-1}(x) = -x$, $x \geq 0$

The inverse of $f(x) = -x$, $x \leq 0$ is

$$f^{-1}(x) = -x, \quad x \geq 0$$

Exercise 6.4

Determine whether the following functions f and g are inverse to each other.

a) $f(x) = x + 3$ and $g(x) = x - 3$

b) $f(x) = -x - 4$ and $g(x) = 4 - x$

c) $f(x) = 2x + 3$ and $g(x) = x - \frac{3}{2}$

Sol To show f and g are inverse to each other, we need to check ① $f(g(x)) = x$

② $g(f(x)) = x$.

a) $f(x) = x+3$, $g(x) = x-3$

① $f(g(x)) = g(x)+3 = (x-3)+3 = x$ and

② $g(f(x)) = f(x) - 3 = (x+3) - 3 = x$

Then, by ① ②, f and g are inverse to each other.

b) $f(x) = -x-4$, $g(x) = 4-x$

① $f(g(x)) = -(g(x)) - 4 = -(4-x) - 4$

$= -4 + x - 4$

$= -8 + x \neq x$

since $f(g(x)) \neq x$, then f and g are NOT inverse to each other

c) $f(x) = 2x+3$, $g(x) = x - \frac{3}{2}$

① $f(g(x)) = 2(g(x)) + 3 = 2\left(x - \frac{3}{2}\right) + 3$

$= 2x - 3 + 3 = 2x \neq x$

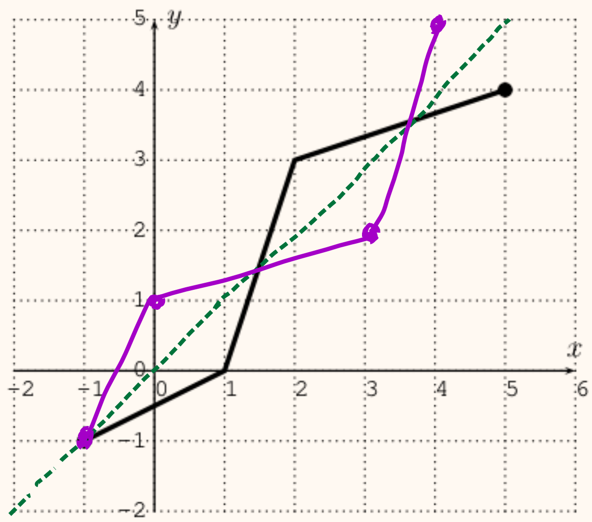
Since $f(g(x)) = 2x \neq x$, then

f and g are NOT inverse to each other.

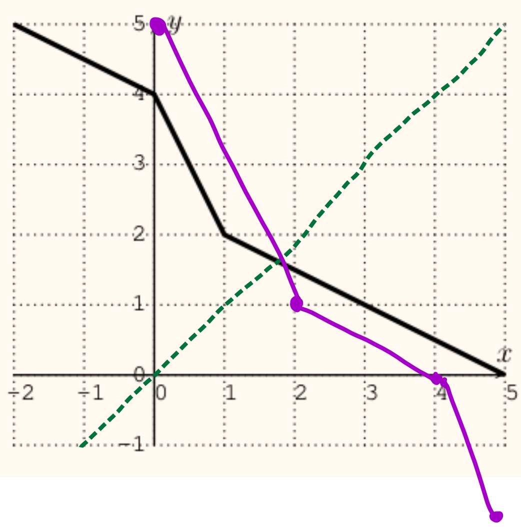
Exercise 6.5

Draw the graph of the inverse of the function given below.

✓ a)



b)



d) $f(x) = \sqrt{x}$