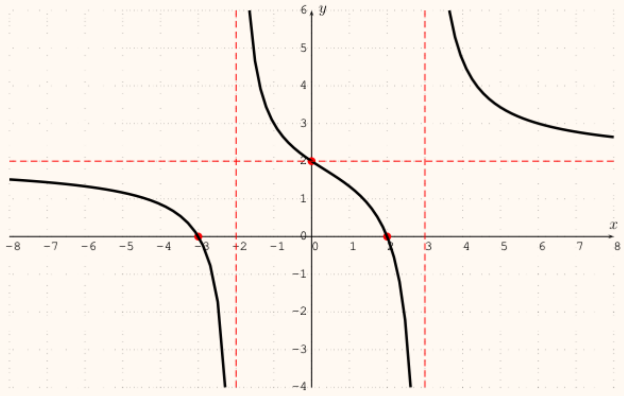
Mat 1375 HW 11

Exercise 11.1

Below are the graphs of rational functions whose numerators and denominators are polynomials of degree 2. All intercepts and asymptotes are at integer values, indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

a)



① x-intercepts: $(2,0)$, $(-3,0)$

② y-intercept: $(0,2)$

③ V.A. $x=3$, $x=-2$

④ H.A. $y=2$

Formula: $f(x)=\frac{p(x)}{q(x)}$, $\deg(p)=2$, $\deg(q)=2$

$q(x)$: By ③, we know V.A. occurs at roots of $q(x)$

since the roots of $q(x)$ are $x=3$ and $x=-2$, then

$$q(x)=(x-3)(x+2)$$

$p(x)$: By ①, we know $f(2)=\frac{p(2)}{q(2)}=0$ and $f(-3)=\frac{p(-3)}{q(-3)}=0$.

Then $p(2)=0$, $p(-3)=0$ which means

2 and -3 are roots of $p(x)$

$\Rightarrow$ $p(x)$ has factors $(x-2)$ and $(x+3)$

Let $p(x)=C\cdot(x-2)(x+3)$

$f(x)=\frac{p(x)}{q(x)}=\frac{C(x-2)(x+3)}{(x-3)(x+2)}$. By ②, we know $f(0)=2$

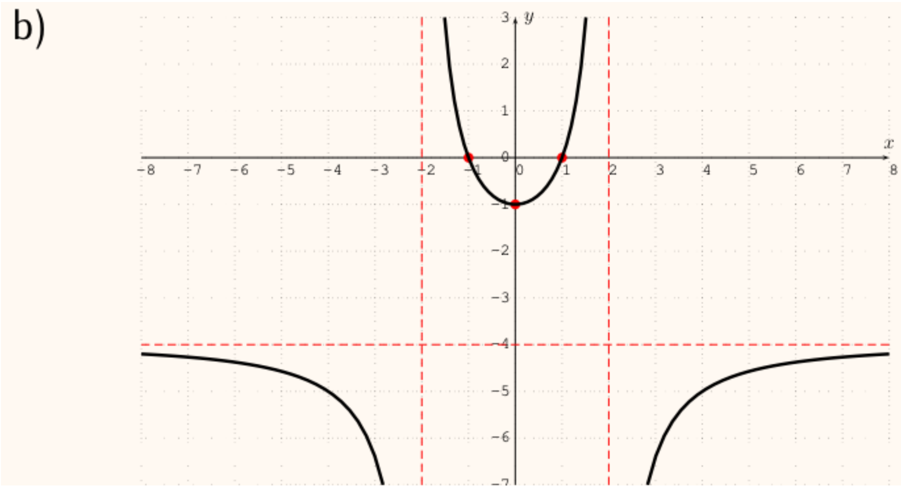
$$\Rightarrow 2=f(0)=\frac{C(0-2)(0+3)}{(0-3)(0+2)}=\frac{C(-2)\cdot 3}{(-3)\cdot 2}=\frac{-6\cdot C}{-6}=C$$

$$\Rightarrow \boxed{f(x)=\frac{2(x-2)(x+3)}{(x-3)(x+2)}}$$

(Also recheck it By ④, $\deg(p)=2=\deg(q)$

then H.A. is $y=\frac{2x^2}{x^2}=2$ V)

b)



① x-intercepts: $(1,0)$, $(-1,0)$

② y-intercept: $(0,-1)$

③ V.A. $x=2$, $x=-2$

④ H.A. $y=-4$.

⑤ $q(x)$: By ③. We know V.A. occurs at roots of $q(x)$

Since the roots of $q(x)$ are $x=2$ and $x=-2$, then

$$q(x) = (x-2)(x+2)$$

⑥ $p(x)$: By ①. We know $f(1) = \frac{p(1)}{q(1)} = 0$ and $f(-1) = \frac{p(-1)}{q(-1)} = 0$.

Then $p(1) = 0$, $p(-1) = 0$ which means

1 and -1 are roots of $p(x)$

$\Rightarrow$ $p(x)$ has factors $(x-1)$ and $(x+1)$

Let $p(x) = C(x-1)(x+1)$

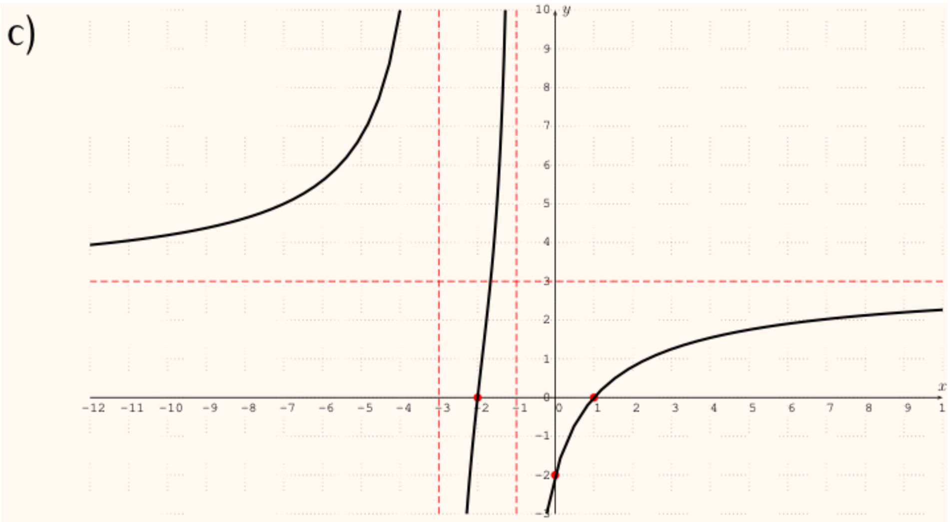
⑦ $f(x) = \frac{p(x)}{q(x)} = \frac{C(x-1)(x+1)}{(x-2)(x+2)}$, By ② We know $f(0) = -1$

$$\Rightarrow -1 = f(0) = \frac{C(0-1)(0+1)}{(0-2)(0+2)} = \frac{C(-1)}{(-2)\cdot 2} = \frac{-C}{-4} = \frac{C}{4}$$

$$\Rightarrow -1 = \frac{C}{4} \Rightarrow C = -4 \Rightarrow f(x) = \frac{-4(x-1)(x+1)}{(x-2)(x+2)}$$

(Also check by ④ H.A.: $\deg(p(x)) = 2 = \deg(q(x)) \Rightarrow$
H.A is $y = \frac{-4}{1} = -4$ ✓)

c)



① x-intercepts: $(1,0)$, $(-2,0)$

② y-intercept $(0,-2)$

③ V.A $x=-1$, $x=-3$

④ H.A. $y=3$.

⑤ $q(x)$: By ③. We know V.A. occurs at roots of $q(x)$

Since the roots of $q(x)$ are $x=-1$ and $x=-3$, then

$$q(x)=(x+1)(x+3)$$

⑥ $p(x)$: By ①, we know $f(1)=\frac{p(1)}{q(1)}=0$ and $f(-2)=\frac{p(-2)}{q(-2)}=0$.

Then $p(1)=0$, $p(-2)=0$ which means

1 and -2 are roots of $p(x)$

$\Rightarrow$ $p(x)$ has factors $(x-1)$ and $(x+2)$

Let $p(x)=C(x-1)(x+2)$

⑦ $f(x)=\frac{p(x)}{q(x)}=\frac{C(x-1)(x+2)}{(x+1)(x+3)}$, By ② we know $f(0)=-2$

$$\Rightarrow -2=f(0)=\frac{C\cdot(0-1)\cdot(0+2)}{(0+1)(0+3)}=\frac{C\cdot(-1)\cdot 2}{1\cdot 3}=\frac{-2C}{3}$$

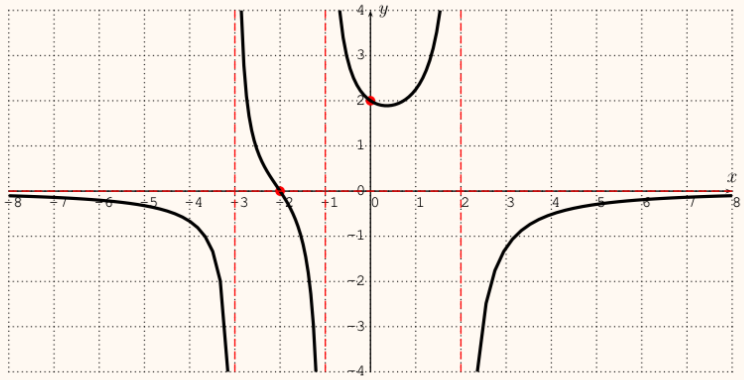
$$\Rightarrow -2=\frac{-2C}{3} \Rightarrow C=3 \qquad \Rightarrow f(x)=\frac{3(x-1)(x+2)}{(x+1)(x+3)}$$

(Also check by ④ H.A.: $\deg(p(x))=2=\deg(q(x)) \Rightarrow$
H.A is $y=\frac{3}{1}=3$ ✓)

Exercise 11.2

Below are the graphs of rational functions whose numerators are polynomials of degree 1 and whose denominators are polynomials of degree 3. All intercepts and asymptotes are at integer values indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

a)



① X-intercept: $(-2,0)$

② y-intercept: $(0,2)$

③ V.A. $x=2, x=-1, x=-3$

④ H.A. $y=0$

⑤ $q(x)$: By ③. We know V.A. occurs at roots of $q(x)$

Since the roots of $q(x)$ are $x=2, x=-1$, and $x=-3$, then

$$q(x) = (x-2)(x+1)(x+3)$$

⑥ $p(x)$: By ①. We know $f(-2) = \frac{p(-2)}{q(-2)} = 0$, then $p(-2)=0$ means

-2 is a root of $p(x) \Rightarrow p(x)$ has factor $(x+2)$

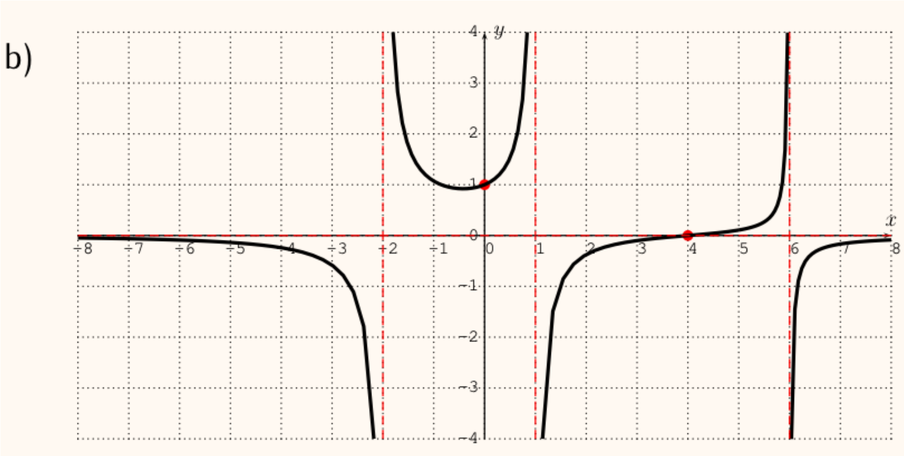
Let $p(x) = C \cdot (x+2)$

⑦ $f(x) = \frac{p(x)}{q(x)} = \frac{C(x+2)}{(x-2)(x+1)(x+3)}$, By ②, we know $f(0)=2$

$$\Rightarrow 2 = f(0) = \frac{C\cdot(0+2)}{(0-2)(0+1)(0+3)} = \frac{2\cdot C}{(-2)\cdot 1\cdot (3)} = \frac{2C}{-6}$$

$$\Rightarrow 2 = \frac{2C}{-6} \Rightarrow C=-6 \Rightarrow f(x) = \frac{-6(x+2)}{(x-2)(x+1)(x+3)}$$

b)



① x-intercept: $(4,0)$

② y-intercept: $(0,1)$

③ V.A. $x=6,\ x=1,\ x=-2$

④ H.A. $y=0$.

⑤ $q(x)$: By ③. We know V.A. occurs at roots of $q(x)$

Since the roots of $q(x)$ are $x=6, x=1$, and $x=-2$, then

$$q(x)=(x-6)(x-1)(x+2)$$

⑥ $p(x)$: By ①. We know $f(4)=\frac{p(4)}{q(4)}=0$, then $p(4)=0$ means

4 is a root of $p(x) \Rightarrow p(x)$ has factor $(x-4)$

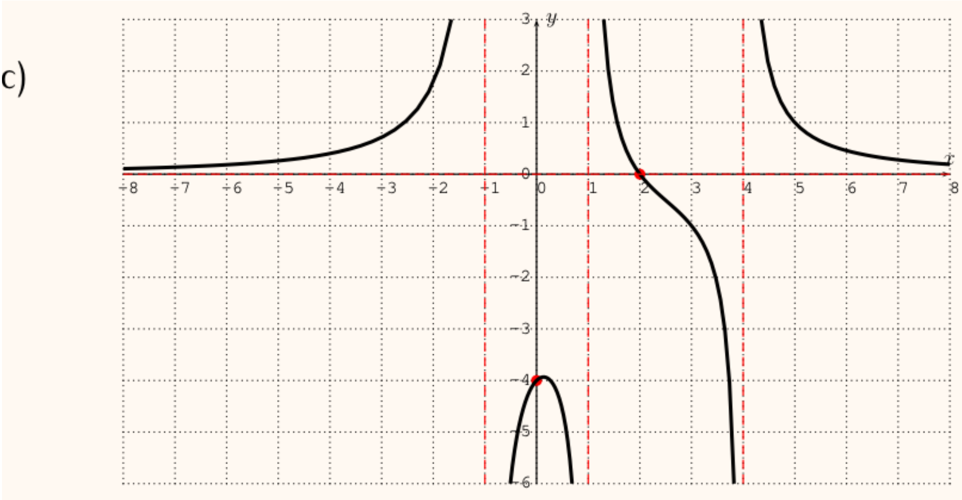
Let $p(x)=C\cdot(x-4)$

⑦ $f(x)=\frac{p(x)}{q(x)}=\frac{C(x-4)}{(x-6)(x-1)(x+2)}$. By ②, we know $f(0)=1$.

$$\Rightarrow 1=f(0)=\frac{C(0-4)}{(0-6)(0-1)(0+2)}=\frac{(-4)C}{(-6)(-1)(2)}=\frac{-4C}{12}$$

$$\Rightarrow 1=\frac{-4C}{12} \Rightarrow C=-3 \Rightarrow f(x)=\frac{-3(x-4)}{(x-6)(x-1)(x+2)}$$

c)



① x-intercept: $(2,0)$

② y-intercept: $(0,-4)$

③ V.A. $x=4, x=1, x=-1$

④ H.A. $y=0$.

⑤ $q(x)$: By ③. we know V.A. occurs at roots of $q(x)$

Since the roots of $q(x)$ are $x=4$, $x=1$, and $x=-1$, then

$$q(x) = (x-4)(x-1)(x+1)$$

⑥ $p(x)$: By ①. we know $f(-4) = \dfrac{p(-4)}{q(-4)} = 0$, then $p(-4)=0$ means

-4 is a root of $p(x) \Rightarrow p(x)$ has factor $(x+4)$

Let $p(x) = C(x+4)$

⑦ $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{C(x+4)}{(x-4)(x-1)(x+1)}$. By ②, we know $f(0) = -4$

$$\Rightarrow -4 = f(0) = \frac{C(0+4)}{(0-4)(0-1)(0+1)} = \frac{4C}{(-4)(-1)\cdot 1} = \frac{4C}{4} = C$$

$\Rightarrow C = -4$, Then $f(x) = \dfrac{-4(x+4)}{(x-4)(x-1)(x+1)}$

Exercise 11.3

Find the domain of each rational function below. Identify the removable discontinuities and find their x- and y-coordinates.

a) $f(x) = \frac{(x-3)(x-4)}{(x+5)(x-4)}$ b) $f(x) = \frac{3(x+2)(x-5)}{(x+3)(x-5)}$

c) $f(x) = \frac{7(x-2)}{(x+3)(x-2)(x-6)}$ d) $f(x) = \frac{x^2+6x+8}{x^2+x-12}$

Sol: (a) $f(x) = \frac{(x-3)(x-4)}{(x+5)(x-4)}$

Domain: All real numbers but $x+5 \neq 0$, $x-4 \neq 0$ ($\Rightarrow x \neq -5, x \neq 4$)

$\Rightarrow x \in (-\infty, -5) \cup (-5, 4) \cup (4, \infty)$

Removable discontinuity: $x-4=0 \Rightarrow x=4$ for $f(x) = \frac{(x-3)}{(x+5)}$

$\Rightarrow f(4) = \frac{(4-3)}{(4+5)} = \frac{1}{9} \Rightarrow$ removable point is $(4, \frac{1}{9})$

(b) $f(x) = \frac{3(x+2)(x-5)}{(x+3)(x-5)}$

Domain: All real numbers but $x+3 \neq 0$, $x-5 \neq 0$ ($\Rightarrow x \neq -3, x \neq 5$)

$\Rightarrow x \in (-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

Removable discontinuity: $x-5=0 \Rightarrow x=5$ for $f(x) = \frac{3(x+2)}{(x+3)}$

$\Rightarrow f(5) = \frac{3(5+2)}{(5+3)} = \frac{3 \cdot 7}{8} = \frac{21}{8} \Rightarrow$ removable point is $(5, \frac{21}{8})$

(c) $f(x) = \frac{7(x-2)}{(x+3)(x-2)(x-6)}$

Domain: All real numbers but $x+3 \neq 0$, $x-2 \neq 0$, $x-6 \neq 0$ ($\Rightarrow x \neq -3, x \neq 2, x \neq 6$)

$\Rightarrow x \in (-\infty, -3) \cup (-3, 2) \cup (2, 6) \cup (6, \infty)$

Removable discontinuity: $x-2=0 \Rightarrow x=2$ for $f(x) = \frac{7}{(x+3)(x-6)}$

$\Rightarrow f(2) = \frac{7}{(2+3)(2-6)} = \frac{7}{5 \cdot (-4)} = \frac{-7}{20} \Rightarrow$ removable point is $(2, -\frac{7}{20})$

d) $f(x)=\frac{x^2+6x+8}{x^2+x-12}=\frac{(x+2)(x+4)}{(x-3)(x+4)}$

Domain: All real numbers but $x\neq -3$, $x\neq -4$,

$\Rightarrow x\in(-\infty,-4)\cup(-4,-3)\cup(-3,\infty)$

Removable discontinuity: $x+4=0 \Rightarrow x=-4$ for $f(x)=\frac{x+2}{x-3}$

$\Rightarrow f(-4)=\frac{-4+2}{-4-3}=\frac{-2}{-7}=\frac{2}{7} \Rightarrow$ removable point is $(4,\frac{2}{7})$

Exercise 11.4

Find the slant asymptote of the rational function.

a) $f(x)=\frac{2x^3+9x^2-20x-21}{2x^2-3x-4}$ b) $f(x)=\frac{2x^3-13x^2+35x-26}{x^2-4x+6}$

Sol: Let $p(x)=2x^3+9x^2-20x-21$, $q(x)=2x^2-3x-4$, $f(x)=\frac{p(x)}{q(x)}$.

Let $g(x), r(x)$ be two polynomials. such that

$$p(x)=q(x)\cdot g(x)+r(x).$$

By long division, we have

$$p(x)=q(x)(x+6)+2x+3$$

$$\begin{array}{r|l} & x+6 \\ \hline 2x^2-3x-4 & 2x^3+9x^2-20x-21 \\ & -(2x^3-3x^2-4x) \\ \hline & 12x^2-16x-21 \\ & -(12x^2-18x-24) \\ \hline & 2x+3 \end{array}$$

$\frac{2x^3}{2x^2}=x$

$\frac{12x^2}{2x^2}=6$

$$\Rightarrow f(x)=\frac{p(x)}{q(x)}=\frac{q(x)(x+6)+2x+3}{q(x)}$$

$$=(x+6)+\frac{2x+3}{2x^2-3x-4}$$

$\Rightarrow y=x+6$ is a slant asymptote of f.