Mat 1375 HW10

## Exercise 10.1

Find the domain, the vertical asymptotes, and removable discontinuities of the functions:

a) $f(x) = \frac{2}{x-2}$ b) $f(x) = \frac{x^2+2}{x^2-6x+8}$

c) $f(x) = \frac{3x+6}{x^3-4x}$ d) $f(x) = \frac{(x-2)(x+3)(x+4)}{(x-2)^2(x+3)(x-5)}$

e) $f(x) = \frac{x-1}{x^3-1}$ f) $f(x) = \frac{2}{x^3-2x^2-x+2}$

**Sol** a) $f(x) = \frac{2}{x-2}$

Domain: All real numbers but $x \neq 2$ (since $x=2$ makes $f$ undefined)

$\Rightarrow \boxed{(-\infty, 2) \cup (2, \infty)}$

Vertical Asymptote(s): The V.A. is located at the zero of denominator of $f$: $x-2=0 \Rightarrow x=2 \Rightarrow$ $\boxed{\text{V.A. is the line } x=2}$

No removable discontinuities.

b) $f(x) = \frac{x^2+2}{x^2-6x+8} = \frac{x^2+2}{(x-2)(x-4)}$

Domain: All real numbers but $x \neq 2$ and $x \neq 4$ (since $\begin{matrix} x=2 \\ x=4 \end{matrix}$ makes $f$ undefined)

$\Rightarrow \boxed{(-\infty, 2) \cup (2, 4) \cup (4, \infty)}$

V.A.: The V.A. is located at the zero of denominator of $f$.

$x-2=0$ and $x-4=0 \Rightarrow x=2$ and $x=4$

$\Rightarrow \boxed{\text{V.A. are } x=2 \text{ and } x=4}$

c) $f(x) = \frac{3x+6}{x^3-4x} = \frac{3x+6}{x(x+2)(x-2)} = \frac{3\boxed{(x+2)}}{x\boxed{(x+2)}(x-2)}$

Domain: All real number but $x \neq 0$, $x \neq -2$, $x \neq 2$

$\Rightarrow \boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)}$

Removable discontinuity: $x+2=0 \Rightarrow \boxed{x=-2}$

V.A.: The V.A. is located at the zero of denominator of $f$:

$x=0$ and $x-2=0 \Rightarrow x=0$ and $x=2$

$\Rightarrow$ $\boxed{\text{V.A. are the line } x=0,\ x=2.}$

d) $f(x) = \dfrac{(x-2)(x+3)(x+4)}{(x-2)^2(x+3)(x-5)} = \dfrac{\boxed{(x-2)}\boxed{(x+3)}(x+4)}{\boxed{(x-2)}(x-2)\boxed{(x+3)}(x-5)}$

Domain: All real numbers but $x \neq 2,\ x \neq -3,\ x \neq 5$

$\Rightarrow x \in (-\infty,-3) \cup (-3,2) \cup (2,5)$

V.A: After cancelation, $f(x) = \dfrac{x+4}{(x-2)(x-5)}$ $\Rightarrow$ $x-2=0,\ x-5=0$

$\Rightarrow x=2,\ x=5$ $\Rightarrow$ V.A. are line $x=2$ and $x=5$

Removable discontinuity: $x+3=0 \Rightarrow x=-3$

e) $f(x) = \dfrac{x-1}{x^3-1} = \dfrac{\boxed{x-1}}{\boxed{(x-1)}(x^2+x+1)}$

Domain: All real numbers but $x \neq 1$ $(x^2+x+1>0)$

$\Rightarrow x \in (-\infty, 1) \cup (1,\infty)$

V.A.: $f(x) = \dfrac{1}{x^2+x+1}$ $\Rightarrow$ NO Vertical Asymptote.

Removable discontinuity: $x-1=0 \Rightarrow x=1$

f) $f(x) = \dfrac{2}{x^3-2x^2-x+2} = \dfrac{2}{(x-2)(x+1)(x-1)}$

Domain: All real numbers but $x \neq -1,\ x \neq 1,\ x=2$ $\Rightarrow$

$x \in (-\infty,-1) \cup (-1,1) \cup (1,2) \cup (2,\infty)$

V.A.: The V.A. is located at the zero of denominator of $f$ $\Rightarrow$

$x-2=0,\ x+1=0,\ x-1=0 \Rightarrow x=2,\ x=-1,\ x=1$

$\Rightarrow$ V.A. are lines $x=2,\ x=-1,\ x=1$.

No Removable discontinuity.

## Exercise 10.2

Find the horizontal asymptotes of the functions:

a) $f(x) = \frac{8x^2+2x+1}{2x^2+3x-2}$ b) $f(x) = \frac{1}{(x-3)^2}$

c) $f(x) = \frac{x^2+3x+2}{x-1}$ d) $f(x) = \frac{12x^3-4x+2}{-3x^3+2x^2+1}$

Sol: a) Let $p(x) = 8x^2+2x+1$, $q(x) = 2x^2+3x-2 = (x+2)(2x-1)$, $f(x) = \frac{p(x)}{q(x)}$

Since $\deg(p) = 2 = \deg(q)$, then

H.A. is the line $y = \frac{\text{leading coeff of } p}{\text{leading coeff. of } q} = \frac{8}{2} \Rightarrow \boxed{y=4}$

b) Let $p(x) = 1$, $q(x) = (x-3)^2 = x^2-6x+9$, $f(x) = \frac{p(x)}{q(x)}$

Since $\deg(p) = 0 < \deg(q) = 2$, then H.A. is the line $\boxed{y=0}$

c) Let $p(x) = x^2+3x+2 = (x+1)(x+2)$, $q(x) = x-1$, $f(x) = \frac{p(x)}{q(x)}$

Since there is NO common factor of $p(x)$ and $q(x)$, and $\deg(p(x)) = 2 < \deg(q(x)) = 1$, then there is NO H.A.

d) Let $p(x) = 12x^3-4x+2$, $q(x) = -3x^3+2x^2+1 = (x-1)(-3x^2-x-1)$, $f(x) = \frac{p(x)}{q(x)}$

$$\begin{array}{r|rrrr} 1 & -3 & 2 & 0 & 1 \\ & & -3 & -1 & -1 \\ \hline & -3 & -1 & -1 & \boxed{0} \end{array}$$

Since there is NO common factor of $p(x)$ and $q(x)$, and $\deg(p(x)) = 3 = \deg(q(x)) = 3$, then

H.A. is the line $y = \frac{\text{leading coeff of } p}{\text{leading coeff. of } q} = \frac{12}{-3} = -4 \Rightarrow \boxed{y=-4}$

## Exercise 10.3

Find the $x$- and $y$-intercepts of the functions:

a) $f(x) = \frac{x-3}{x-1}$ b) $f(x) = \frac{x^3-4x}{x^2-8x+15}$

c) $f(x) = \frac{(x-3)(x-1)(x+4)}{(x-2)(x-5)}$ d) $f(x) = \frac{x^2+5x+6}{x^2+2x}$

Sol (a) $f(x) = \frac{x-3}{x-1}$

x-intercept(s): $0 = f(x) = \frac{x-3}{x-1} \Rightarrow x-3=0 \Rightarrow x=3 \Rightarrow (3,0)$

y-intercept (s): $f(0) = \frac{0-3}{0-1} = \frac{-3}{-1} = 3 \Rightarrow (0,3)$

(b) $f(x) = \frac{x^3-4x}{x^2-8x+15} = \frac{x(x+2)(x-2)}{(x-3)(x-5)}$

x-intercepts: $0 = f(x) = \frac{x(x+2)(x-2)}{(x-3)(x-5)} \Rightarrow x(x+2)(x-2)=0$

$\Rightarrow x=0,\ x+2=0,\ x-2=0 \Rightarrow x=0,\ x=-2,\ x=2$

$\Rightarrow (0,0),\ (-2,0),\ (2,0)$

y-intercept: $f(0) = \frac{(0)^3-4(0)}{(0)^2-8\cdot(0)+15} = \frac{0}{15} = 0 \Rightarrow (0,0)$

(c) $f(x) = \frac{(x-3)(x-1)(x+4)}{(x-2)(x-5)}$

x-intercepts: $0 = f(x) = \frac{(x-3)(x-1)(x+4)}{(x-2)(x-5)} \Rightarrow (x-3)(x-1)(x+4)=0$

$\Rightarrow x-3=0,\ x-1=0,\ x+4=0 \Rightarrow x=3,\ x=1,\ x=-4$

$\Rightarrow (3,0),\ (1,0),\ (-4,0)$

y-intercept: $f(0) = \frac{(0-3)(0-1)(0+4)}{(0-2)(0-5)} = \frac{(-3)\cdot(-1)\cdot 4}{(-2)\cdot(-5)} = \frac{12}{10} = \frac{6}{5}$

$\Rightarrow (0, \frac{6}{5})$

d) $f(x) = \frac{x^2+5x+6}{x^2+2x} = \frac{\boxed{(x+2)}(x+3)}{x\boxed{(x+2)}} = \frac{x+3}{x}$ but $x \neq -2$

x-intercept: $0 = f(x) = \frac{x+3}{x} \Rightarrow x+3=0 \Rightarrow x=-3 \Rightarrow (-3,0)$

y-intercept: x cannot be 0 since $x=0$ is a V.A.
$\Rightarrow$ NO y-intercept.

## Exercise 10.4

Sketch a complete graph of the function $f$. To this end, calculate the domain of $f$, the horizontal and vertical asymptotes, the removable singularities, the $x$- and $y$-intercepts of the function, and graph the function with the graphing calculator.

✓a) $f(x) = \frac{7x+2}{3x-5}$ ✓b) $f(x) = \frac{x^2-x-2}{x^2+2x-3}$

Sol: a) $f(x) = \dfrac{7x+2}{3x-5}$ , Let $p(x) = 7x+2$ , $q(x) = 3x-5$

Domain: All real numbers but $3x-5 \neq 0$, $(x \neq \frac{5}{3})$
$\Rightarrow x \in (-\infty, \frac{5}{3}) \cup (\frac{5}{3}, \infty)$

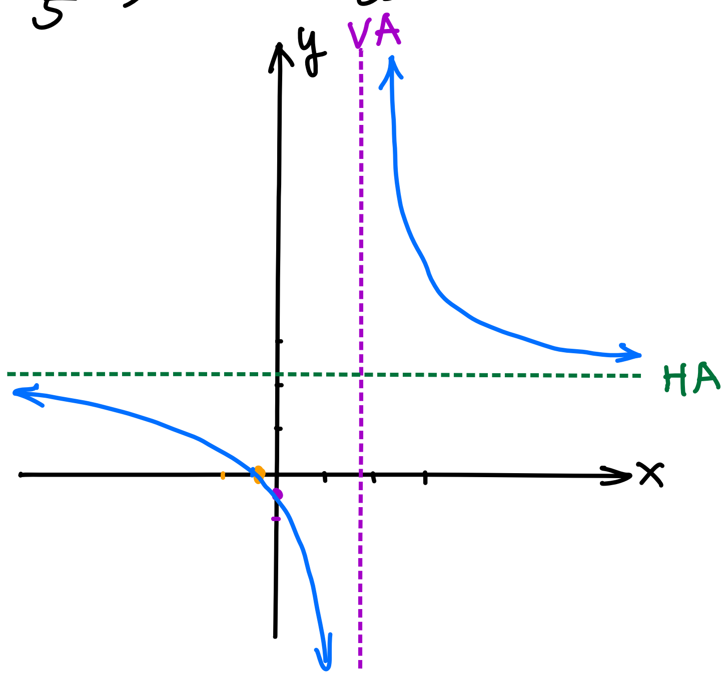
H.A. Since $\deg(p(x)) = 1 = \deg(q(x))$, then
H.A. is the line $y = \dfrac{\text{leading coeff of } p}{\text{leading coeff. of } q} = \dfrac{7}{3} \Rightarrow y = \dfrac{7}{3}$

V.A.: V.A. occurs at $3x-5=0 \Rightarrow x = \frac{5}{3} \Rightarrow$
V.A. is the line $x = \frac{5}{3}$

x-intercept: $0 = f(x) = \dfrac{7x+2}{3x-5} \Rightarrow 7x+2=0 \Rightarrow x = -\frac{2}{7} \Rightarrow (-\frac{2}{7}, 0)$

y-intercept: $f(0) = \dfrac{7 \cdot 0 + 2}{3 \cdot 0 - 5} = \dfrac{2}{-5} = -\dfrac{2}{5} \Rightarrow (0, -\frac{2}{5})$

Graph



b) $f(x) = \dfrac{x^2-x-2}{x^2+2x-3} = \dfrac{(x-2)(x+1)}{(x+3)(x-1)}$ $\left( \dfrac{x^2+2x-3\ -3x+1}{x^2+2x-3} = 1 + \dfrac{-3x+1}{x^2+2x-3} \right)$

Domain: All real numbers but $x \neq -3$, $x \neq 1$ $\Rightarrow x \in (-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

H.A. Let $p(x) = x^2 - x - 2$, $q(x) = x^2 + 2x - 3$.

Since there is NO factor of p and q, and

$\deg(p(x)) = 2 = \deg(q)$, then

H.A. is the line $y = \dfrac{\text{leading coeff of } p}{\text{leding coeff. of } q} = \dfrac{1}{1} \Rightarrow y = 1$

V.A. V.A. occurs at $(x+3)(x-1) = 0 \Rightarrow x+3=0$, $x-1=0$

V.A. are lines $x = -3$, $x = 1$.

x-intercept: $0 = f(x) = \dfrac{(x-2)(x+1)}{(x+3)(x-1)} \Rightarrow (x-2)(x+1) = 0 \Rightarrow x-2=0$, $x+1=0$

$\Rightarrow x = 2$, $x = -1 \Rightarrow \boxed{(2,0)}$, $\boxed{(-1,0)}$

y-intercept: $f(0) = \dfrac{(0)^2-(0)-2}{(0)^2+2(0)-3} = \dfrac{-2}{-3} = \dfrac{2}{3} \Rightarrow \boxed{\left(0, \tfrac{2}{3}\right)}$

Graph: