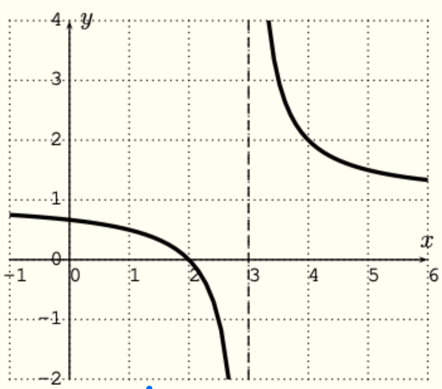
MAT 1375, Classwork9, Fall2024

ID:______________________________ Name:______________________________

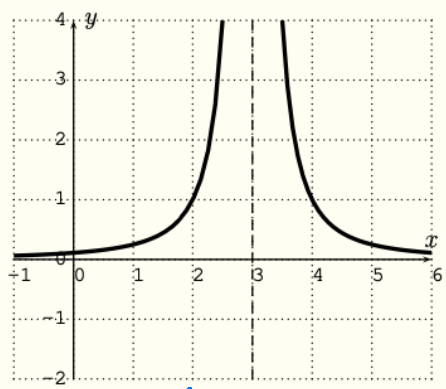
1. The definition of a **Vertical Asymptote**:

The line $\boldsymbol{x} = \boldsymbol{a}$ is a Vertical asymptote of the graph of a function f if $f(x)$ **increases or decreases without bound** as $\boldsymbol{x}$ **approaches** $\boldsymbol{a}$.

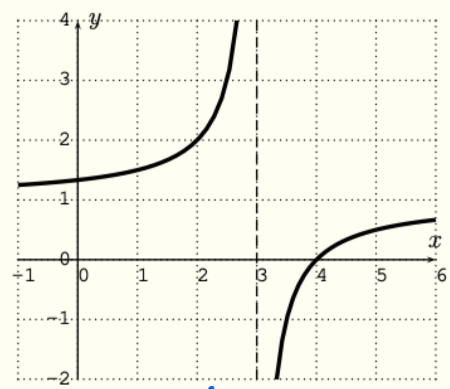
$f(x) = \frac{x-2}{x-3}$



$f(x) = \frac{1}{(x-3)^2}$



$f(x) = \frac{(x-3)(x-4)}{(x-3)^2}$



As $x \to 3^+$, $f(x) \to \infty$; As $x \to 3^+$, $f(x) \to \infty$ As $x \to 3^+$, $f(x) \to -\infty$

As $x \to 3^-$, $f(x) \to -\infty$. As $x \to 3^-$, $f(x) \to -\infty$ As $x \to 3^-$, $f(x) \to \infty$

2. How to locate Vertical Asymptotes: Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function.

If $p(x)$ and $q(x)$ have no common factors,

and $\boldsymbol{a}$ is a **zero** of $q(x)$ which makes $f(x)$ undefined,

then $x = a$ is a vertical asymptote of the graph of $f(x)$.

If $\boldsymbol{a}$ is a **zero** of both $p(x)$ and $q(x)$ ($p(a) =$ 0, $q(a) =$ 0.) which means $x-a$ is the common factor of $p(x)$ and $q(x)$, then there is a removable discontinuity/jump/singularity at $x = a$.

3. Find the vertical asymptotes of the graph of each rational function:

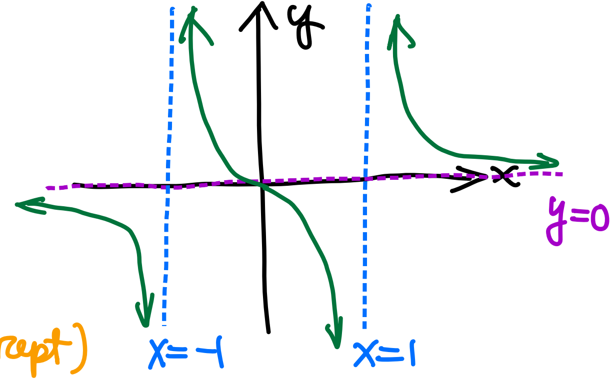
a) $f(x) = \frac{x}{x^2-1}$ b) $g(x) = \frac{x-1}{x^2-1}$ c) $h(x) = \frac{x-1}{x^2+1}$

3(a) $f(x) = \frac{x}{x^2-1}$. Let $p(x) = x$, $q(x) = x^2-1$, $f(x) = \frac{p(x)}{q(x)}$.

$q(x) = x^2-1 = (x+1)(x-1) = 0$

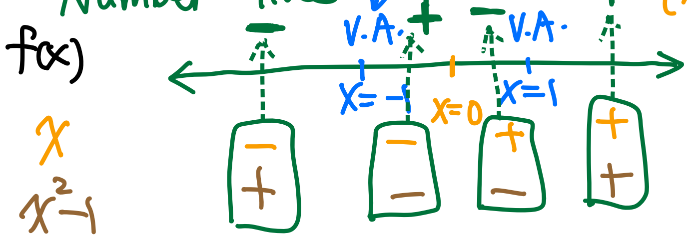
$\Rightarrow x=1$ and $x=-1$ are zeros of $q(x)$ (which make f undefined)

$\Rightarrow$ the line $x=1$ and $x=-1$ are vertical asymptotes of f.



Number line

(x=0 is x-intercept)



(Also, $deg(p)=1$, $deg(q)=2$

$\Rightarrow$ $y=0$ is a horizontal asymptote)

3(b) $g(x) = \frac{x-1}{x^2-1}$, Let $p(x)=x-1$, $q(x)=x^2-1$, $g(x)=\frac{p(x)}{q(x)}$.

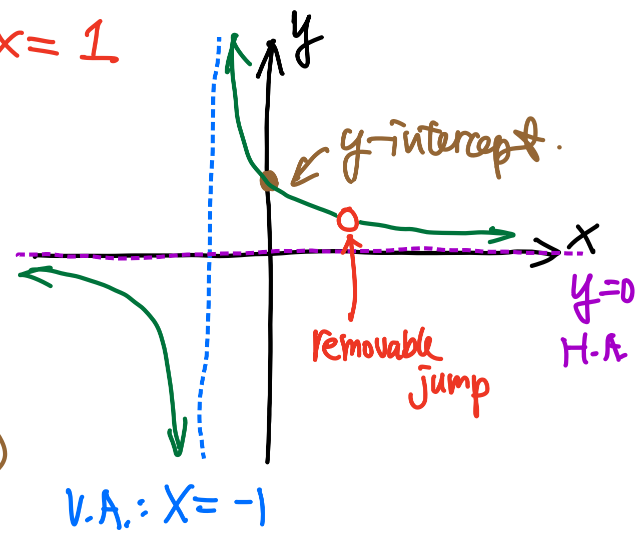
Since $q(x)=x^2-1=(x+1)(x-1) \Rightarrow g(x)=\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{x+1}$, but $x \neq 1$

Then we have only one vertical asymptote: $x+1=0 \Rightarrow x=-1$

with a $\boxed{\text{removable jump}}$ at $x=1$

Also, since $deg(p)=1 < deg(q)=2$, then

$y=0$ is a horizontal asymptote.

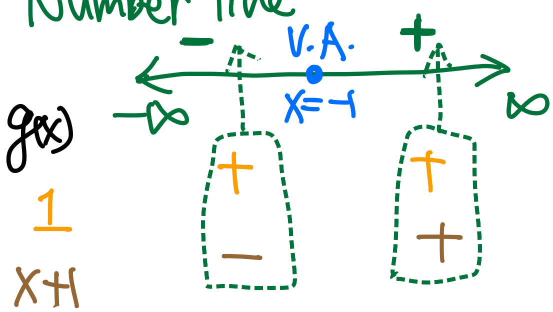


x-intercept: $g(x)=\frac{1}{x+1}$ has no x-intercept.

y-intercept: $g(x)=\frac{1}{x+1}$ has a y-intercept:

$(g(0)=1) \longrightarrow (0,1)$

Number line



3(c) $h(x)=\dfrac{x-1}{x^2+1}$. Let $p(x)=x-1$, $q(x)=x^2+1$, $h(x)=\dfrac{p(x)}{q(x)}$

Since $q(x)=x^2+1\geq 1$ (can not be zero), then there is

NO Vertical Asymptote (V.A.) for $h(x)$.

Since $\deg(p)=1<\deg(q)=2$, then we have

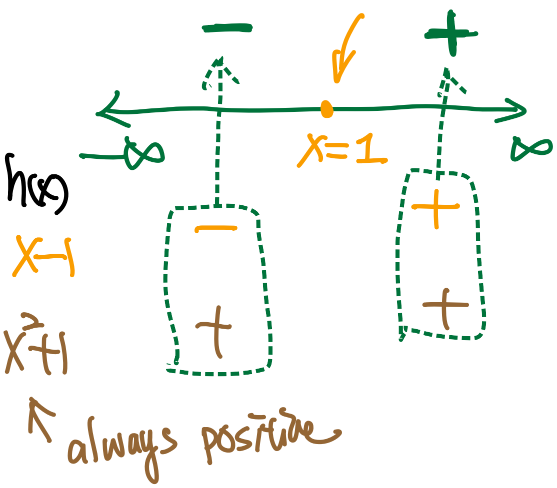
$y=0$ is a horizontal Asymptote (H.A.)

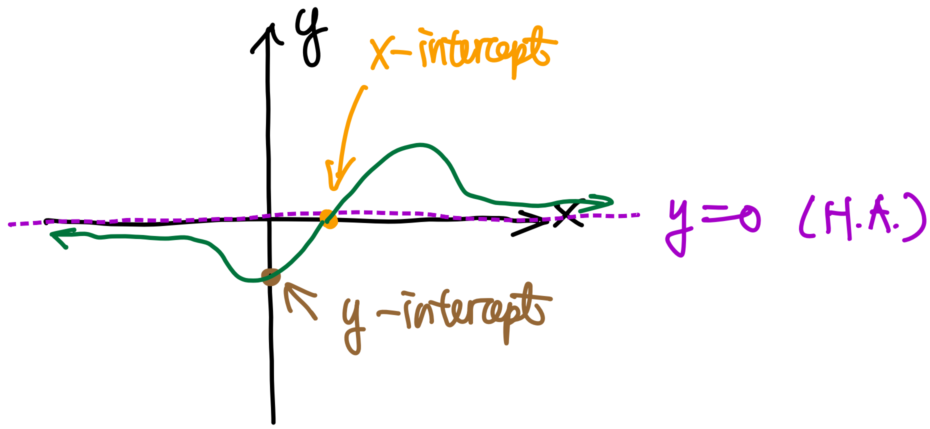
x-intercept (as $y=0$) $\Rightarrow \dfrac{x-1}{x^2+1}=0 \Rightarrow x-1=0 \Rightarrow x=1$

$(1,0)$ is a x-intercept.

y-intercept (as $x=0$) $\Rightarrow h(0)=\dfrac{0-1}{0^2+1}=-1 \Rightarrow (0,-1)$ is a y-intercept.

Number line





4. The definition of a **Horizontal Asymptote**:

The line $\boldsymbol{y} = \boldsymbol{b}$ is a horizontal asymptote the graph of a function f if $f(x)$ approaches $\boldsymbol{b}$ as $\boldsymbol{x}$ **increases** or **decreases without bound**.

5. What is the difference of Vertical Asymptote and Horizontal Asymptote?

Vertical asymptote occurs at $x=c$ when $\boxed{f(x) \to \pm\infty}$

Horizontal asymptote occurs at $\boxed{x \longrightarrow \pm\infty}$ and $f(x)$ approaches to a number.

6. How to locate Horizontal Asymptotes: Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function given by

$$f(x) = \frac{p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0}{q_m x^m + q_{m-1}x^{m-1} + \cdots + q_1 x + q_0}, \; p_n \neq 0, q_m \neq 0.$$

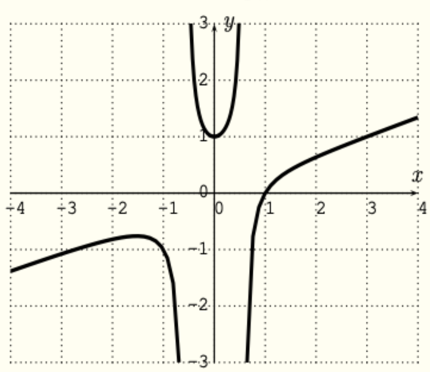
The degree of the numerator is n. The degree of the denominator is m.

1) If $n > m$, the graph of f has NO horizontal asymptote.

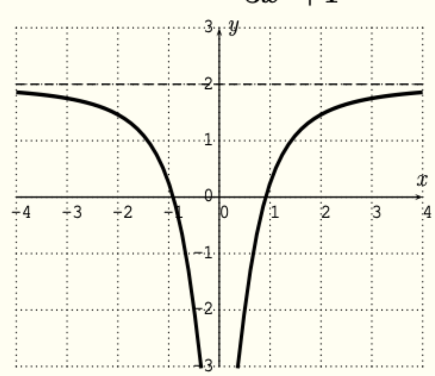
2) If $n = m$, the line $y = \frac{p_n}{q_m}$ (which is the ration of two leading coefficients) is the horizontal asymptote of the graph of f.

3) If $n < m$, the X-axis (which is $y=0$) is the horizontal asymptote of the graph of f.

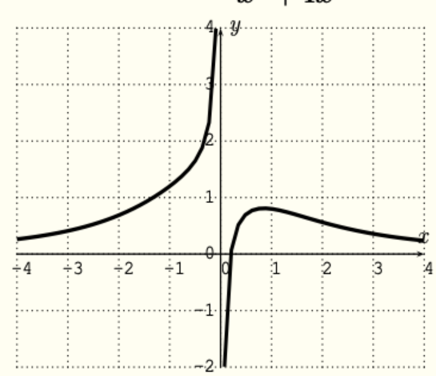
$f(x) = \frac{x^3-1}{3x^2-1}$



$f(x) = \frac{6x^2-5}{3x^2+1}$



$f(x) = \frac{5x-1}{x^3+4x}$



deg($p(x)$) $>$ deg($q(x)$) | deg($p(x)$) $=$ deg($q(x)$) | deg($p(x)$) $<$ deg($q(x)$)

No horizontal asymptote | $y = \frac{6}{3} \Rightarrow y=2$ | $y=0.$