Math 1451, Honor Calculus Practice 10, Spring 2016.

April 6, 2016

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1. Set up the integral for

$$\iiint_E x^2 dV,$$

where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

Cylinder coordinate

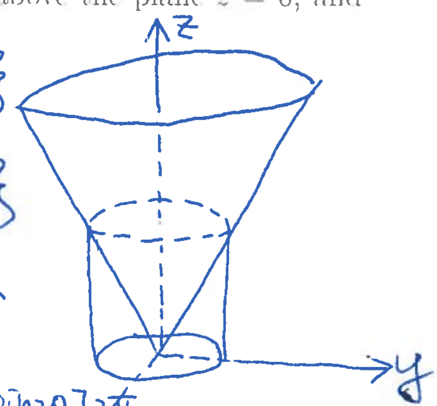
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

$$E = \{(x,y,z) \mid 0 \le x^2+y^2 \le 1,\ 0 \le z \le \sqrt{4x^2+4y^2}\}$$

$$= \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 2r\}$$

Then $\iiint_E x^2 dV = \int_0^1 \int_0^{2\pi} \int_0^{2r} r^2\cos^2\theta\, r\, dz\, d\theta\, dr$

$$= \int_0^1 \int_0^{2\pi} 2r^4 \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta\, dr = \left[\frac{2}{5}r^5\Big|_0^1\right]\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{2}{5}\cdot\pi.$$



2. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ where $a > 0$ if S has constant density K.

$$\boxed{E = \{(x,y,z) \mid 0 \le x^2+y^2 \le \tfrac{a}{4},\ 4x^2+4y^2 \le z \le a\}}$$

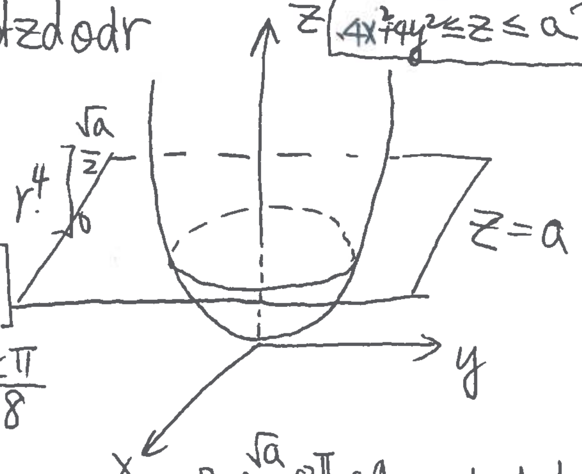
$$m = \iiint_E K\, dz\, dx\, dy = K\int_0^{\frac{\sqrt{a}}{2}} \int_0^{2\pi} \int_{4r^2}^{a} r\, dz\, d\theta\, dr$$

By Cylinder coord.

$$= K\cdot 2\pi \int_0^{\frac{\sqrt{a}}{2}} r(a-4r^2)\, dr = 2K\pi\cdot\left[a\frac{r^2}{2} - r^4\right]_0^{\frac{\sqrt{a}}{2}}$$

$$= 2K\pi\left[\frac{a^2}{8} - \frac{a^2}{16}\right] = \frac{K\pi}{8}$$

$$\bar{x} = \frac{1}{m}\iiint_E Kx\, dz\, dx\, dy = 0,$$



$$\bar{y} = \frac{1}{m}\iiint_E Ky\, dz\, dx\, dy = 0,\quad \bar{z} = \frac{1}{m}\iiint_E Kz\, dz\, dx\, dy = \frac{8}{\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_0^{2\pi}\int_{4r^2}^{a} zr\, dz\, dr\, d\theta$$

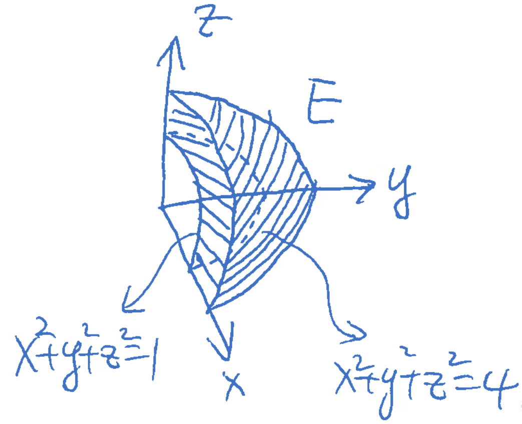
$$= \frac{8}{\pi}\cdot\frac{1}{2}\cdot 2\pi \int_0^{\frac{\sqrt{a}}{2}} r(a^2 - (4r^2)^2)\, dr = \frac{2}{3}a.$$

3. Set up the integral

$$\iiint_E z\,dV$$

where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

Using spherical coordinates



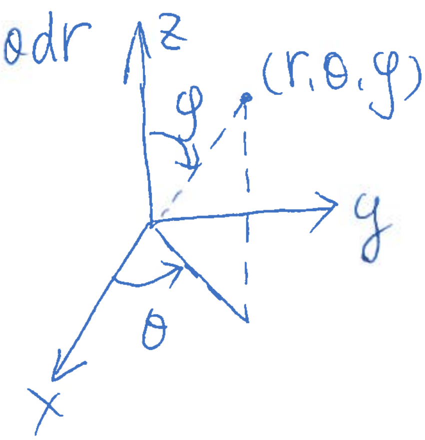
let $x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

Then we have

$$\iiint_E z\,dV = \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r\cos\varphi \cdot r^2\sin\varphi \, d\varphi\, d\theta\, dr$$



$$= \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^3 \cos\varphi \sin\varphi \, d\varphi\, d\theta\, dr$$

$$= \left[\int_1^2 r^3\,dr\right]\left[\int_0^{\frac{\pi}{2}} \cos\varphi\sin\varphi\,d\varphi\right]\left[\int_0^{\frac{\pi}{2}} d\theta\right]$$

$$= \left[\frac{r^4}{4}\Big|_1^2\right] \cdot \left[\frac{\sin^2\varphi}{2}\Big|_0^{\frac{\pi}{2}}\right] \cdot \frac{\pi}{2} = \frac{15}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{15}{16}\pi.$$