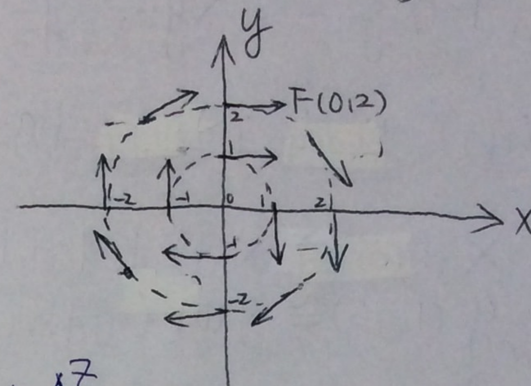
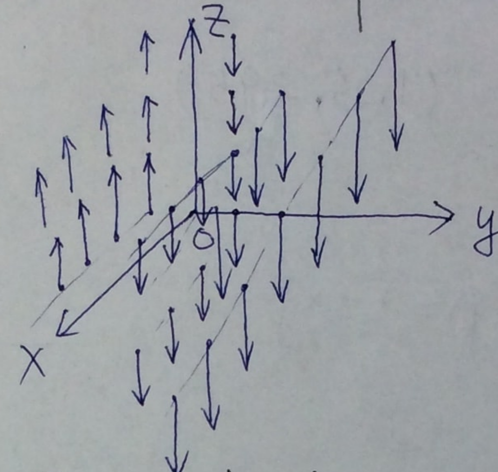
Honors Calculus, Math 1451 – HW 7 (II) – solutions

§ 16.1

6. Given $F(x,y) = \dfrac{y\vec{i} - x\vec{j}}{\sqrt{x^2+y^2}}$



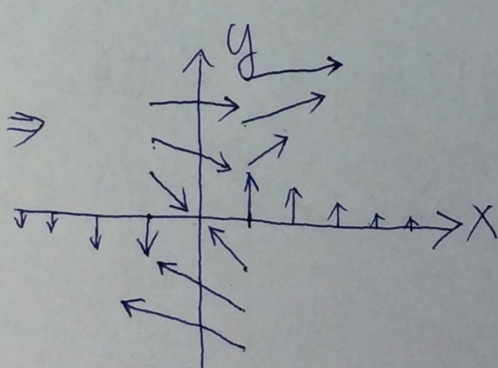
8. Given $F(x,y,z) = -y\vec{k}$



12. Given $F(x,y) = \langle 1, \sin y \rangle$ $\Rightarrow$ x-direction is always positive

$\Rightarrow$ $\underline{\text{IV}}$

14. Given $F(x,y) = \langle y, \frac{1}{x} \rangle \Rightarrow$



$\Rightarrow$ $\underline{\text{III}}$

24. Given $f(x,y,z) = x\cos\left(\frac{y}{z}\right)$

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \left\langle \cos\left(\frac{y}{z}\right), -\frac{x}{z}\sin\left(\frac{y}{z}\right), +\frac{yx}{z^2}\sin\left(\frac{y}{z}\right) \right\rangle$$

34. Given $F(x,y) = \langle xy-2, y^2-10 \rangle$. So if a particle at $(1,3)$ as $t=1$, it has velocity $F(1,3) = \langle 1, -1 \rangle$

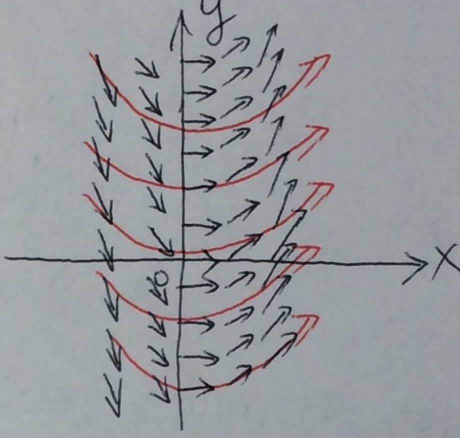
So $x(1.05) \cong x(1) + 1 \cdot (1.05-1) = 1 + 0.05 = 1.05$

$y(1.05) \cong y(1) + (-1) \cdot (1.05-1) = 3 - 0.05 = 2.95$

$\Rightarrow (x,y) = (1.05, 2.95)$.

36.

(a) Given $F(x,y) = \vec{i} + x\vec{j}$



(b) Location: $x(t)\vec{i} + y(t)\vec{j}$.

Velocity: $x'(t)\vec{i} + y'(t)\vec{j} = \vec{i} + x\vec{j} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = x$.

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{1} = x$.

(c) By (b). $\frac{dy}{dx} = x \Rightarrow \int \frac{dy}{dx}\, dx = \int x\, dx = \frac{x^2}{2} + x(0)$

and $x(0) = 0$. $\Rightarrow$ $y = \frac{x^2}{2}$ is an equation of the path the particle follows.

§16.2

4. $\int_C x\sin y\, ds$ where C is the line segment from (0,3) to (4,6)

$\Rightarrow (0,3)+((4,6)-(0,3))t$

$=(4t, 3+3t)\ ,\quad 0\le t\le 1.$

$\int_0^1 4t\,(\sin(3+3t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$=\int_0^1 4t\cdot\sin(3+3t)\cdot\sqrt{4^2+3^2}\,dt = 20\int_0^1 t\sin(3+3t)\,dt$

$=20\left(-\frac{t}{3}\cos(3+3t)-\frac{t}{9}\sin(3+3t)\right)\Big|_0^1$

$=20-\frac{1}{3}\cos(6)-\frac{1}{9}\sin(6).$

t	$\sin(3+3t)$	$+$
1	$\frac{-\cos(3+3t)}{3}$	$-$
0	$\frac{-\sin(3+3t)}{9}$	$+$
		$-$

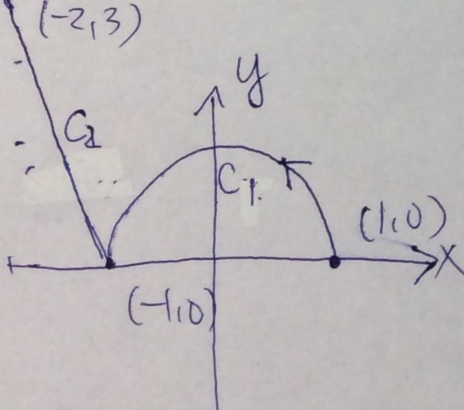
6. $\int_C xe^y\,dx$, C is the curve $x=e^y$ from (1,0) to (e,1).

$=\int_0^1 e^t\cdot e^t\cdot e^t\,dt$

$=\int_0^1 e^{3t}\,dt=\frac{e^{3t}}{3}\Big|_0^1=\frac{e^3}{3}-\frac{1}{3}.$

let $y(t)=t,\ x(t)=e^t$
where $0\le t\le 1$
$\Rightarrow dx=e^t dt.$

8. $\int_C \sin(x)\,dx+\cos(y)\,dy$ where C consists of the top of $x^2+y^2=1$ from (1,0) to (-1,0) and line segment from (-1,0) to (-2,3).



For C_1, let $x(t)=\cos(t)$, $y(t)=\sin(t)$
for $0\le t\le\pi$

For C_2, the segment is $(-1,0)+((-2,3)-(-1,0))t$
$=(-1-t, 3t)$, $\Rightarrow x(t)=-1-t,\ y(t)=3t$
$0\le t\le 1$

Then we have

$$\int_{C_1} \sin(x)dx + \cos(y)dy + \int_{C_2} \sin(x)dx + \cos(y)dy$$

$$= \int_0^{\pi} \left[\sin(\cos(t))\cdot(-\sin(t)) + \cos(\sin(t))\cdot\cos(t)\right]dt + \int_0^1 \sin(-1-t)\cdot(-dt) + \cos(3t)\cdot 3dt$$

$$= -\cos(\cos(t)) + \sin(\sin(t))\Big|_0^{\pi} + \left[-\cos(-1-t) + \sin(3t)\right]_0^1$$

$$= -\cos(-1) + \sin(0) + \cos(1) - \sin(0) - \cos(-2) + \sin(3) + \cos(-1) + \sin(0)$$

$$= -\cos(1) + \cos(1) - \cos(2) + \cos(1) = \cos(1) - \cos(2).$$

12. $\int_C (2x+9z)\,ds$ where C is: $x=t,\ y=t^2,\ z=t^3,\ 0\le t\le 1.$

$$= \int_0^1 (2t+9t^3)\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^1 (2t+9t^3)\cdot\sqrt{1+4t^2+9t^4}\,dt$$

let $u = 1+4t^2+9t^4$

$du = 8t+36t^3$

$\Rightarrow \frac{du}{4} = 2t+9t^3$

$\int \sqrt{u}\,\frac{du}{4} = \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}$

$$= \frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}}{3}\,(1+4t^2+9t^4)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{1}{6}\left[14\sqrt{14}-1\right]$$

18. For C_1, we have $\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_a^b \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt$

where $\vec{r}_1(t)$ is the point on C_1 and $\vec{r}_1{}'(t)$ is the tangent vector of C_1 at time t.

Based on the given vector field, the angles between the vectors of the vector field and $\vec{r}_1{}'(t)$ of C_1 are less than $\frac{\pi}{2}$.

So $\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)$ is always positive

Then $\int_{C_1} \vec{F} \cdot d\vec{r}$ will be positive.

For C_2, we have $\int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_c^d \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt$

where $\vec{r}_2(t)$ is the point on C_2 and $\vec{r}_2{}'(t)$ is the tangent vector of C_2 at time t.

Based on the given vector field, the angles between the vectors of the vector field and $\vec{r}_2{}'(t)$ of C_2 are more than $\frac{\pi}{2}$.

So $\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)$ is alway negative. Then $\int_{C_2} \vec{F} \cdot d\vec{r}$ is negative

22. $\int_C \vec{F}(x,y,z)\cdot d\vec{r}$ where $\vec{F}(x,y,z)=z\vec{i}+y\vec{j}-x\vec{k}$ and

$\vec{r}(t)=t\vec{i}+\sin(t)\vec{j}+\cos(t)\vec{k}$, $0\le t\le\pi$

$\Rightarrow \vec{r}'(t)=\vec{i}+\cos(t)\vec{j}-\sin(t)\vec{k}$

$$=\int_0^{\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$=\int_0^{\pi}\langle\cos(t),\sin(t),-t\rangle\cdot\langle 1,\cos(t),-\sin(t)\rangle\,dt$$

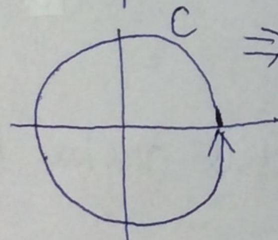
$$=\int_0^{\pi}\cos(t)+\sin(t)\cos(t)+\underline{t\sin(t)}\,dt$$

t	$\sin(t)$	$+$
1	$-\cos(t)$	$-$
0	$-\sin(t)$	$+$
		$-$

$$=-\sin(t)+\frac{\sin^2(t)}{2}+\sin(t)-t\cos(t)\Big|_0^{\pi}$$

$$=0+0+0-\pi(-1)+0=\pi.$$

32. (a) Given force field on a particle: $F(x,y)=x^2\vec{i}+xy\vec{j}$ and the trace of this particle



$\Rightarrow r(t)=2\cos(t)\vec{i}+2\sin(t)\vec{j}$

$r'(t)=-2\sin(t)\vec{i}+2\cos(t)\vec{j}$

$$W=\int_C\vec{F}(x,y)\cdot d\vec{r}=\int\langle 4\cos^2(t),4\cos(t)\sin(t)\rangle\cdot\langle-2\sin(t),2\cos(t)\rangle\,dt$$

$$=\int -8\cos^2(t)\sin(t)+8\cos^2(t)\sin(t)\,dt=0$$

(b). Check the force field, the direction of $\vec{F}(x,y)$ is always perpendicular to the trace of particle.

34. Thin wire's shape $x^2+y^2=a^2$. $x\geq 0, y\geq 0$

$\Rightarrow x=a\cos(t),\ y=a\sin(t).\ 0\leq t\leq \frac{\pi}{2}.$

Given densiey function of this wire be $\rho(x,y)=kxy$,

we have $C=(a\cos(t), a\sin(t)),\ 0\leq t\leq\frac{\pi}{2}.$

$$\text{Mass}=\int_C \rho(x,y)\,ds=\int_0^{\frac{\pi}{2}} k\cdot a\cos(t)\cdot a\sin(t)\cdot\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\,dt$$

$$=\int_0^{\frac{\pi}{2}} a^2k\cos(t)\sin(t)\sqrt{(a\sin(t))^2+(a\cos(t))^2}\,dt$$

$$=\int_0^{\frac{\pi}{2}} a^2k\cos(t)\sin(t)\cdot a\,dt=a^3k\left.\frac{\sin^2(t)}{2}\right|_0^{\frac{\pi}{2}}=\frac{a^3k}{2}$$

and

$$\bar{x}=\frac{1}{\text{Mass}}\int_C x\,\rho(x,y)\,ds=\frac{1}{M}\int_0^{\frac{\pi}{2}} a\cos(t)\cdot ka\cos(t)\cdot a\sin(t)\cdot a\,dt$$

$$=\frac{1}{M}\cdot a^4k\int_0^{\frac{\pi}{2}}\cos^2(t)\sin(t)\,dt=\frac{a^4k}{M}\left[-\left.\frac{\cos^3(t)}{3}\right|_0^{\frac{\pi}{2}}\right]=\frac{a^4k}{3M}$$

$$=\frac{a^4k}{3}\cdot\frac{2}{a^3k}=\frac{2}{3}a.$$

$$\bar{y}=\frac{1}{M}\int y\,\rho(x,y)\,ds=\frac{1}{M}\int_0^{\frac{\pi}{2}} a\sin(t)\cdot ka\cos(t)\cdot a\sin(t)\cdot a\,dt$$

$$=\frac{a^4k}{M}\int_0^{\frac{\pi}{2}}\cos(t)\sin^2(t)\,dt=\frac{a^4k}{M}\cdot\left.\frac{\sin^3(t)}{3}\right|_0^{\frac{\pi}{2}}=\frac{a^4k}{3M}=\frac{2}{3}a$$

42. Given $\vec{F}(\vec{r}(t)) = \frac{K\vec{r}}{|\vec{r}|^3}$ for $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

and a segment from $(2,0,0)$ to $(2,1,5)$.

$\Rightarrow \vec{r}(t) = (2,0,0) + ((2,1,5)-(2,0,0))t = (2,t,5t)$. $0 \le t \le 1$

$\Rightarrow \vec{r}'(t) = \langle 0,1,5 \rangle$ and $|\vec{r}| = \sqrt{4+t^2+25t^2} = \sqrt{4+26t^2}$

Then

$$W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_0^1 K \frac{\langle 2,t,5t \rangle}{(\sqrt{4+26t^2})^3} \cdot \langle 0,1,5 \rangle\,dt$$

$$= K \int_0^1 \frac{26t}{(\sqrt{4+26t^2})^3}\,dt$$

let $u = 4+26t^2 \Rightarrow du = 52t$

$\Rightarrow \frac{du}{2} = 26t$

$\Rightarrow \int \frac{1}{u^{\frac{3}{2}}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-\frac{3}{2}}\,du$

$= \frac{1}{2}(-2)u^{-\frac{1}{2}}$

$= -u^{-\frac{1}{2}}$

$$= K \cdot \left(-\frac{1}{\sqrt{4+26t^2}}\right)\Big|_0^1$$

$$= K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right)$$

48. Since C is a circle with radius r, we have $C: r\cos(\theta)\vec{i} + r\sin(\theta)\vec{j}$

$0 \le \theta \le 2\pi$.

So we have $\int_C \vec{B} \cdot d\vec{r} = \int_0^{2\pi} |B| \langle -\sin(\theta), \cos\theta \rangle \cdot \langle -r\sin(\theta), r\cos(\theta) \rangle\, d\theta$

Since $\vec{B}$ is tangent to cures of any circle on the plane, so.
$\vec{B} = |\vec{B}| \langle -\sin(\theta), \cos\theta \rangle$

$$= \int_0^{2\pi} |\vec{B}| (r\sin^2\theta + r\cos^2\theta)\,d\theta = |\vec{B}| \int_0^{2\pi} r\,d\theta = 2\pi r|\vec{B}|$$

So $\mu_0 I = 2\pi r|\vec{B}| \Rightarrow |\vec{B}| = \frac{\mu_0 I}{2\pi r}$

§16.3

4. Given $\vec{F}(x,y) = e^x\cos(y)\,\vec{i} + e^x\sin(y)\,\vec{j}$.

Let $P(x,y) = e^x\cos(y)$, $Q(x,y) = e^x\sin(y)$.

Then $\frac{\partial P}{\partial y} = -e^x\sin(y)$, $\frac{\partial Q}{\partial x} = e^x\sin(y)$

since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so $\vec{F}(x,y)$ is NOT conservative.

6. Given $\vec{F}(x,y) = (3x^2 - 2y^2)\,\vec{i} + (4xy+3)\,\vec{j}$.

Let $P(x,y) = 3x^2 - 2y^2$, $Q(x,y) = 4xy+3$. We have

$\frac{\partial P}{\partial y} = -4y$ and $\frac{\partial Q}{\partial x} = 4y$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so $\vec{F}(x,y)$ is NOT conservative.

8. Given $\vec{F}(x,y) = (xy\cos(xy) + \sin(xy))\,\vec{i} + (x^2\cos(xy))\,\vec{j}$.

Let $P(x,y) = xy\cos(xy) + \sin(xy)$, $Q(x,y) = x^2\cos(xy)$, we have

$\frac{\partial P}{\partial y} = x\cos(xy) - x^2y\sin(xy) + x\cos(xy)$ and

$\frac{\partial Q}{\partial x} = 2x\cos(xy) - x^2y\sin(xy)$.

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so $\vec{F}(x,y)$ is conservative

12. Given $\vec{F}(x,y)=x^2\vec{i}+y^2\vec{j}$ and C be the arc of parabola $y=2x^2$ from $(-1,2)$ to $(2,8)$

(a) To find f such that $\nabla f=\vec{F}$,

first, checking $\vec{F}$ is conservative. let $P(x,y)=x^2$, $Q(x,y)=y^2$.

We have $\frac{\partial P}{\partial y}=0=\frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative,

that is, f exists.

Since $\frac{\partial f}{\partial x}=P(x,y)=x^2$ and $\frac{\partial f}{\partial y}=Q(x,y)=y^2$.

We have $f(x,y)=\int \frac{\partial f}{\partial x}\,dx=\int x^2\,dx=\frac{x^3}{3}+g(y)$

and $y^2=\frac{\partial f(x,y)}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x^2}{3}+g(y)\right)=\frac{\partial g}{\partial y} \Rightarrow g=\frac{y^3}{3}$

Then $f(x,y)=\frac{x^3}{3}+\frac{y^3}{3}$.

(b) By (a), we have

$$\int_C \vec{F}\cdot d\vec{r}=f(2,8)-f(-1,2)=\left(\frac{8}{3}+\frac{8^3}{3}\right)-\left(-\frac{1}{3}+\frac{8}{3}\right)$$

$$=\frac{513}{3}$$

14. Given $F(x,y) = \frac{y^2}{1+x^2}\vec{i} + 2y\arctan(x)\vec{j}$ and the curve C:
$r(t) = t^2\vec{i} + 2t\vec{j}$, $0 \le t \le 1$ $\Rightarrow$ from $\underset{(t=0)}{(0,0)}$ to $\underset{(t=1)}{(1,2)}$

(a) To find f such that $\nabla f = \vec{F}$. first, checking $\vec{F}$ is conservative. let $P(x,y) = \frac{y^2}{1+x^2}$, $Q(x,y) = 2y\arctan(x)$.

We have $\frac{\partial P}{\partial y} = \frac{2y}{1+x^2} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative, that is, f exists.

Since $\frac{\partial f}{\partial x} = \frac{y^2}{1+x^2}$ and $\frac{\partial f}{\partial y} = 2y\arctan(x)$. Then

$f(x,y) = \int \frac{\partial f}{\partial y}\,dy = y^2\arctan(x) + g(x)$ and

$$\frac{y^2}{1+x^2} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(y^2\arctan(x) + g(x)\right) = \frac{y^2}{1+x^2} + \frac{\partial g}{\partial x} \Rightarrow \frac{\partial g}{\partial x} = 0$$

$\Rightarrow g(x)$ is a constant c.

$\Rightarrow f(x,y) = \frac{y^2}{1+x^2} + c$.

(b) By (a), we have

$$\int_C \vec{F}\cdot d\vec{r} = f(1,2) - f(0,0) = \left(\frac{4}{2}+c\right) - \left(\frac{0}{1}+c\right)$$
$$= 2.$$

20. Given $\int_C (1-ye^{-x})dx + e^{-x}dy$ and C be any path from $(0,1)$ to $(1,2)$.

To show this integral is independent of path,

let $F(x,y) = (1-ye^{-x})\vec{i} + e^{-x}\vec{j}$, we have

$$\int_C (1-ye^{-x})dx + e^{-x}dy = \int_C \vec{F}(x,y)\cdot d\vec{r}$$

So if $\vec{F}$ is conservative, there is f such that $\nabla f = \vec{F}$.

and $\int_C \vec{F}\cdot d\vec{r} = f(x_1,y_1) - f(x_2,y_2)$, that is,

the integral is independent of path.

To check $\vec{F}$ is conservative, let $P(x,y) = 1-ye^{-x}$, $Q(x,y) = e^{-x}$

we have $\frac{\partial P}{\partial y} = -e^{-x} = \frac{\partial Q}{\partial x}$, So $\vec{F}$ is conservative.

To find f, we have $\frac{\partial f}{\partial x} = P = 1-ye^{-x}$, $\frac{\partial f}{\partial y} = e^{-x}$

We have $f(x,y) = \int \frac{\partial f}{\partial x}dx = \int 1-ye^{-x}dx = x+ye^{-x}+g(y)$ and

$$e^{-x} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x+ye^{-x}+g(y)) = e^{-x} + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = \text{constant } c$$

$$\Rightarrow f(x,y) = x + e^{-x}y + c$$

Then
$$\int_C \vec{F}(x,y)\cdot d\vec{r} = f(1,2) - f(0,0)$$
$$= (1+2e^{-1}+c) - (0+0+c)$$
$$= 1+\frac{2}{e}$$

24. For a given vector field $\vec{F}(x,y)$. we have

$\vec{F}(x,y)$ is conservative if the line integral $\int_C \vec{F}\cdot d\vec{r} = 0$

for any closed path C.

This given vector field is conservative since for any closed path C in this domain.

34.

(a) Given $\vec{F}(\vec{r}) = \dfrac{c\vec{r}}{|\vec{r}|^3}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

To Find the work done by $\vec{F}$ in moving a particle from P_1 to P_2 where $|P_1| = d_1$, $|P_2| = d_2$.

we have $f(x,y,z) = \dfrac{-c}{|\vec{r}|} = \dfrac{-c}{\sqrt{x^2+y^2+z^2}}$, and

$$W = \int_{\text{From } P_1 \text{ to } P_2} \vec{F}(\vec{r})\cdot d\vec{r} = f(P_2) - f(P_1) = \frac{-c}{|P_2|} - \frac{-c}{|P_1|}$$

$$= c\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

34.

(b) By (a). we have $\vec{F} = -\frac{(mMG)}{|\vec{r}|^3}\vec{r}$ and

$|P_1| = 1.52\times10^8$ (km), $|P_2| = 1.47\times10^8$ (km)

where $m = 5.97\times10^{24}$ kg. $M = 1.99\times10^{30}$ kg, and

$G = 6.67\times10^{-11}$ $N\cdot m^2/kg^2$

Then $W = -mMG\cdot\left(\frac{1}{1.52\times10^8} - \frac{1}{1.47\times10^8}\right) \cong 1.77\times10^{32}$ (J).

(c) By (a), we have $\vec{F} = \frac{\varepsilon q Q}{|\vec{r}|^3}\vec{r}$ and $|P_1| = 10^{-12}$. $|P_2| = 0.5\times10^{-12}$.

where $\varepsilon = 8.985\times10^9$. $q = 1$. $Q = -1.6\times10^{-19}$,

Then

$$W = \varepsilon q Q\left(\frac{1}{10^{-12}} - \frac{1}{0.5\times10^{-12}}\right) \cong 1400 \text{(J)}$$