Honors Calculus, Math 1451, HW7 (I) — solutions

§15.7

4. (a) $(x,y,z) = (2\sqrt{3}, 2, -1) \rightarrow \boxed{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{matrix}} \rightarrow \begin{matrix} 2\sqrt{3} = r\cos\theta \\ 2 = r\sin\theta \\ z = -1 \end{matrix}$

$r^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = (2\sqrt{3})^2 + (2)^2 = 16 \Rightarrow r = 4$

$\cos\theta = \frac{2\sqrt{3}}{4}, \ \sin\theta = \frac{2}{4} \Rightarrow \theta = \frac{\pi}{6}$

$\Rightarrow (r,\theta,z) = (4, \frac{\pi}{6}, -1)$.

(b) $(x,y,z) = (4,-3,2) \rightarrow \boxed{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{matrix}} \rightarrow \begin{matrix} 4 = r\cos\theta \\ -3 = r\sin\theta \\ z = 2 \end{matrix}$

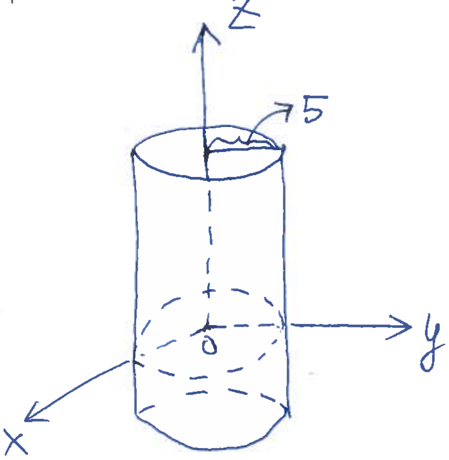
$\Rightarrow r^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = 4^2 + (-3)^2 = 25 \Rightarrow r = 5$.

$\cos\theta = \frac{4}{5}, \ \sin\theta = -\frac{3}{5} \Rightarrow \tan\theta = -\frac{3}{4} \Rightarrow \theta = \tan^{-1}(-\frac{3}{4})$.

$\Rightarrow (r,\theta,z) = (5, \tan^{-1}(-\frac{3}{4}), 2)$

6. $r = 5 \Rightarrow \theta$ and z are arbitrary

$\Rightarrow$ It is a cylinder with radius 5

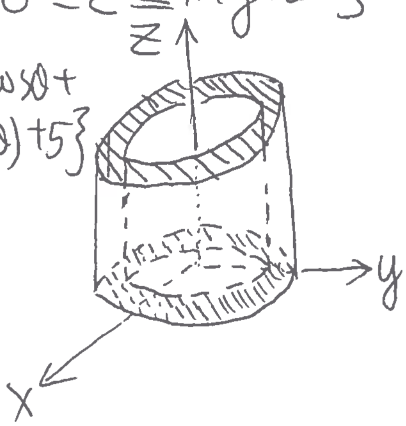


20. Find $\iiint_E x\,dV$ where $E = \{(x,y,z) \mid 4 \le x^2+y^2 \le 9, \ 0 \le z \le x+y+5\}$

$\Rightarrow E = \{(r,\theta,z) \mid 2 \le r \le 3, \ 0 \le \theta \le 2\pi, \ 0 \le z \le r(\cos\theta + \sin\theta) + 5\}$

$\iiint_E x\,dV = \int_2^3 \int_0^{2\pi} \int_0^{r\cos\theta + r\sin\theta + 5} r\cos\theta \cdot r \, dz\, d\theta\, dr$

$= \int_2^3 \int_0^{2\pi} r^2\cos\theta\,(r\cos\theta + r\sin\theta + 5)\, d\theta\, dr$



$$= \int_2^3 \int_0^{2\pi} r^3\cos^2\theta + r^3\cos\theta\sin\theta + 5r^2\cos\theta \, d\theta \, dr$$

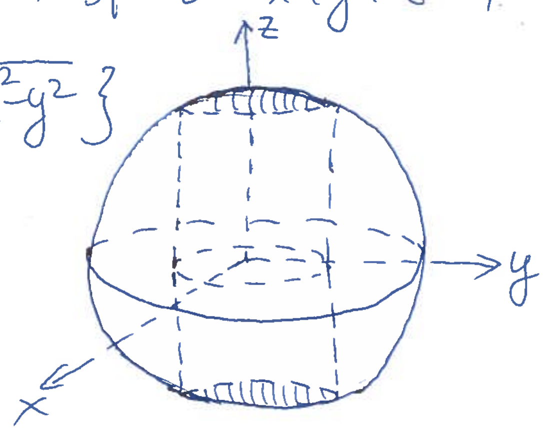
$$= \int_2^3\int_0^{2\pi} r^3\left(\frac{1+\cos 2\theta}{2}\right)d\theta\, dr + \int_2^3\int_0^{2\pi} r^3\cos\theta\sin\theta\, d\theta\, dr + \left[\int_2^3 5r^2 dr\right]\left[\int_0^{2\pi}\cos\theta\, d\theta\right]$$

$$= \left[\frac{1}{2}\cdot\frac{r^4}{4}\Big|_2^3\right]\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} + \left[\frac{r^4}{4}\right]_2^3\left[\cancel{\frac{\sin^2\theta}{2}}\right]_0^{2\pi \; \to 0} + \left[\frac{5r^3}{3}\right]_2^3 \cancel{\left[\sin\theta\right]_0^{2\pi}} \; 0$$

$$= \frac{65}{8}\cdot 2\pi = \frac{65\pi}{4}$$

22. Volume within cylinder $x^2+y^2=1$ and sphere $x^2+y^2+z^2=4$

$$E = \{(x,y,z) \mid 0 \le x^2+y^2 \le 1,\ -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}\}$$



$$\iiint_E 1 \, dV$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$E = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\}$$

$$\int_0^1\int_0^{2\pi}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz\, d\theta\, dr = \int_0^1\int_0^{2\pi} r\left[\sqrt{4-r^2} - \left(-\sqrt{4-r^2}\right)\right] d\theta\, dr$$

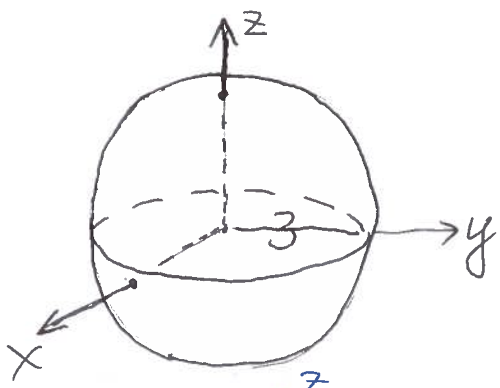
$$= 2\int_0^1\int_0^{2\pi} r\sqrt{4-r^2}\, d\theta\, dr = 2\cdot 2\pi\left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{4\pi}{3}\left[8 - 3\sqrt{3}\right]$$

§15.8.

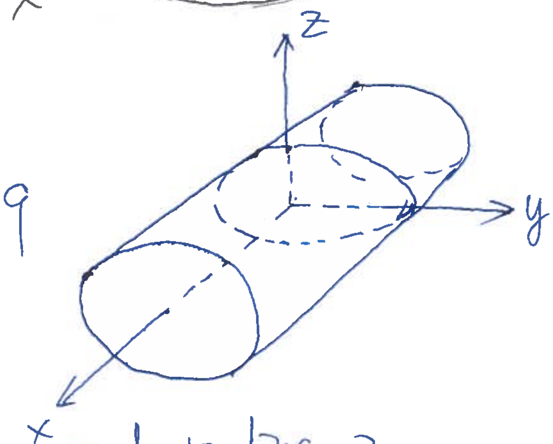
6. $\rho=3 \Rightarrow \theta$ and φ are arbitrary $\Rightarrow$

It is a sphere with radius 3.



8. $\rho^2(\sin^2\phi\sin^2\theta+\cos^2\phi)=9$

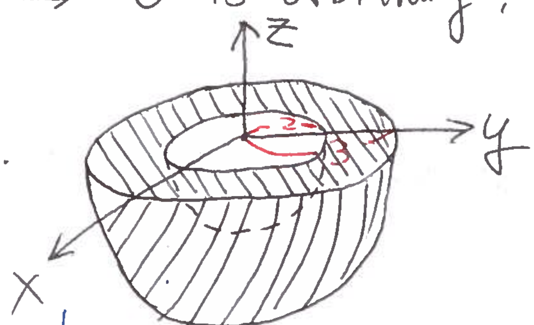
$$\rightarrow \boxed{\begin{array}{l} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{array}} \Rightarrow y^2+z^2=9$$



$\Rightarrow$ It's a cylinder parallel with x-axis and radius 3.

12. $2\le\rho\le3$, $\frac{\pi}{2}\le\phi\le\pi \Rightarrow \theta$ is arbitrary, $0\le\theta\le2\pi$

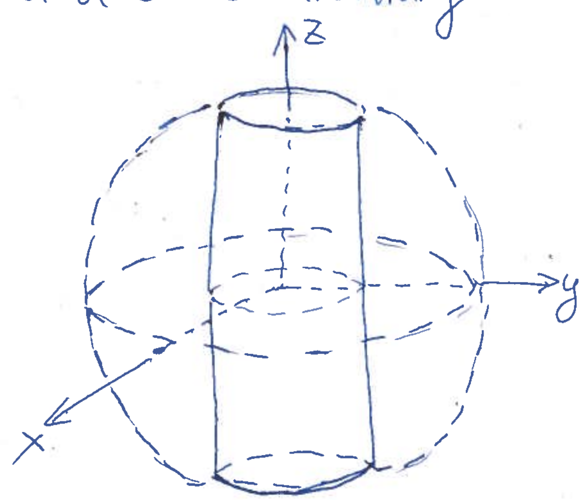
$\Rightarrow$ It is a lower-half shell.



14. $\rho\le2$, $\rho\le\csc\phi \Rightarrow \rho\le\frac{1}{\sin\phi}$ and θ is arbitrary

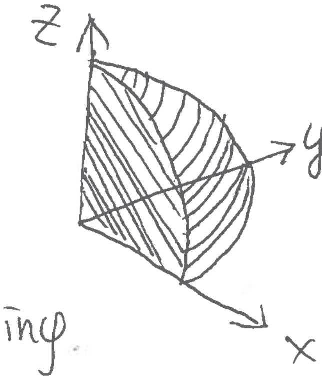
$\Rightarrow \rho\le2$, $\rho\sin\phi\le1$, $0\le\theta\le2\pi$

$\Rightarrow \rho\le2$ $\sqrt{x^2+y^2}\le1$, $0\le\theta\le2\pi$



$\Rightarrow$ It is a cylinder parallel with z-axis and a top and buttom of a radius-2 sphere.

24. $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$ where E is enclosed by $x^2+y^2+z^2=9$ in the first octant.



$E=\{(\rho,\theta,\varphi)\mid 0\le\rho\le 3,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\varphi\le\frac{\pi}{2}\}$

with $x=\rho\sin\varphi\cos\theta$
$y=\rho\sin\varphi\sin\theta$ and $|J(\rho,\theta,\varphi)|=\rho^2\sin\varphi$
$z=\rho\cos\varphi$

$$=\int_0^3\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} e^{\rho}\,\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho=\frac{\pi}{2}\underbrace{\left[\int_0^3\rho^2 e^{\rho}\,d\rho\right]}\left[\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\right]$$

$$=\frac{\pi}{2}\left[e^{\rho}(\rho^2-2\rho+2)\right]_0^3\cdot\left[-\cos\varphi\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}\left[5e^3-2\right]\cdot 1=\frac{(5e^3-2)}{2}\pi$$

u	dv	sign
ρ^2	e^{ρ}	+
2ρ	e^{ρ}	−
2	e^{ρ}	+
0	e^{ρ}	−
		+

34. Solid hemisphere of radius a

$\Rightarrow\{(r,\theta,\varphi)\mid 0\le r\le a,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{2}\}=E$

Density at a point is proportional to its distance from the base $\Rightarrow$ The density function is $D(x,y,z)=K\cdot z$.

$$\text{Mass}=\iiint_E D(x,y,z)\,dV=\int_0^a\int_0^{2\pi}\int_0^{\frac{\pi}{2}}K\cdot\rho\cos\varphi\,\rho^2\sin\varphi\,d\varphi\,d\theta\,dr$$

$$=K\left[\int_0^a\rho^3\,d\rho\right]\left[2\pi\right]\left[\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\,d\varphi\right]$$

$$=K\left[\frac{\rho^4}{4}\Big|_0^a\right]\left[2\pi\right]\left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}}=\frac{K}{4}\cdot a^4\cdot\cancel{2}\pi\cdot\frac{1}{\cancel{2}}=\frac{a^4K}{4}\pi.$$

$$\bar{x} = \frac{1}{M}\iiint_E x\,kz\,dV = \frac{k}{M}\int_0^a\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho\sin\varphi\cos\theta\ \rho\cos\varphi\ \rho^2\sin\varphi\ d\varphi\, d\theta\, d\rho$$

$$= \frac{k}{M}\left[\int_0^a \rho^4 d\rho\right]\left[\cancel{\int_0^{2\pi}\cos\theta\, d\theta} \xrightarrow{} 0\right]\left[\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos\varphi\, d\varphi\right] = 0$$

$$\bar{y} = \frac{1}{M}\iiint_E y\,kz\,dV = \frac{k}{M}\int_0^a\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho\sin\varphi\sin\theta\ \rho\cos\varphi\ \rho^2\sin\varphi\ d\varphi\, d\theta\, d\rho$$

$$= \frac{k}{M}\left[\int_0^a \rho^4 d\rho\right]\left[\cancel{\int_0^{2\pi}\sin\theta\, d\theta} \xrightarrow{} 0\right]\left[\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos\varphi\, d\varphi\right] = 0$$

$$\bar{z} = \frac{1}{M}\iiint_E z\,kz\,dV = \frac{k}{M}\int_0^a\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho^2\cos^2\varphi\ \rho^2\sin\varphi\ d\varphi\, d\theta\, d\rho$$

$$= \frac{k}{M}\left[\int_0^a \rho^4 d\rho\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\frac{\pi}{2}} \cos^2\varphi\sin\varphi\, d\varphi\right]$$

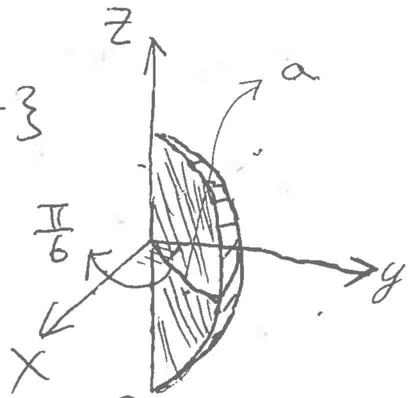
$$= \frac{k}{M}\left[\frac{\rho^5}{5}\Big|_0^a\right]\cdot 2\pi\left[-\frac{\cos^3\varphi}{3}\right]_0^{\frac{\pi}{2}} = \frac{k}{M}\cdot\frac{a^5}{5}\cdot 2\pi\,\frac{1}{3}$$

$$= \cancel{k}\cdot\frac{4}{\cancel{a^4}\cancel{k}\cancel{\pi}}\cdot\frac{a^{\cancel{5}}}{15}\cdot 2\cancel{\pi} = \frac{8a}{15}$$

$\Rightarrow$ center of mass $= (\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{8a}{15})$

36.

$$E = \{(r,\theta,\varphi) \mid 0 \le r \le a,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le \tfrac{\pi}{6}\}$$



$$\iiint_E 1\,dV = \int_0^a \int_0^\pi \int_0^{\frac{\pi}{6}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$= \left[\int_0^a \rho^2 \, d\rho\right]\left[\int_0^{\frac{\pi}{6}} d\theta\right]\left[\int_0^\pi \sin\varphi \, d\varphi\right]$$

$$= \left[\frac{\rho^3}{3}\Big|_0^a\right] \cdot \frac{\pi}{6} \cdot \left[-\cos\varphi\right]_0^\pi$$

$$= \frac{a^3}{3} \cdot \frac{\pi}{6} \cdot (1-(-1)) = \frac{1}{9} a^3 \pi$$