Honors Calculus, Math 1451, HW6, Solutions

§15.2

10. $\int_0^1\int_0^3 e^{x+3y}\,dx\,dy = \int_0^1 e^{3y}dy\cdot\int_0^3 e^x dx$

$= \frac{e^{3y}}{3}\Big|_0^1\cdot e^x\Big|_0^3 = \left(\frac{e^3}{3}-\frac{1}{3}\right)(e^3-1) = \frac{(e^3-1)^2}{3}$

12. $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \int_0^1 \frac{x}{3}(x^2+y^2)^{\frac{3}{2}}\Big|_0^1 dx$

$= \int_0^1 \frac{x}{3}(x^2+1)^{\frac{3}{2}} - \frac{x^4}{3}\,dx = \frac{1}{15}(x^2+1)^{\frac{5}{2}} - \frac{x^5}{15}\Big|_0^1$

$= \frac{1}{15}\left[4\sqrt{2}-1\overset{-}{\cancel{+}}1\right] = \frac{4}{15}\sqrt{2}-\frac{2}{15}$

14. $\int_0^1\int_0^1 \sqrt{s+t}\,ds\,dt = \int_0^1 \frac{2}{3}(s+t)^{\frac{3}{2}}\Big|_0^1 dt = \int_0^1 \frac{2}{3}(t+1)^{\frac{3}{2}}-\frac{2}{3}t^{\frac{3}{2}}dt$

$= \frac{4}{15}(t+1)^{\frac{5}{2}} - \frac{4}{15}t^{\frac{5}{2}}\Big|_0^1 = \frac{4}{15}\cdot 4\sqrt{2} = \frac{16}{15}\sqrt{2}$

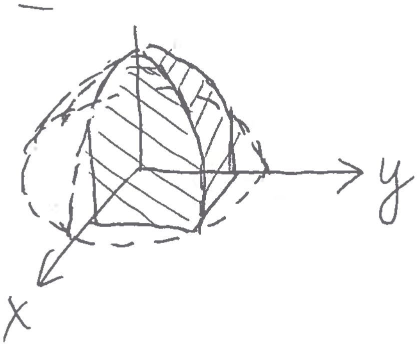
20. $\iint_R \frac{x}{1+xy}\,dA$, $R=[0,1]\times[0,1]$

$= \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \ln(1+xy)\Big|_0^1 dx = \int_0^1 \ln(1+x)\,dx$

$= x\ln(1+x)\overset{-}{\cancel{+}}x + \ln(1+x)\Big|_0^1 = \ln 2 \overset{-}{\cancel{+}} 1 + \ln 2 = \cancel{1}$

$2\ln 2 - 1$

24. $\int_0^1\int_0^1 2-x^2-y^2\,dx\,dy = $

Vol ()

26. Volume under $z=4+x^2-y^2$ and above the square $[-1,1]\times[0,2]$

$$\int_0^2\int_{-1}^1 4+x^2-y^2\,dx\,dy = \int_0^2 4x+\frac{x^3}{3}-y^2x\Big|_{-1}^{1} dy$$

$$= \int_0^2 4+\frac{1}{3}-y^2-\left(-4-\frac{1}{3}+y^2\right)dy = \int_0^2 8+\frac{2}{3}-2y^2\,dy$$

$$= \frac{26}{3}y-\frac{2}{3}y^3\Big|_0^2 = \frac{52}{3}-\frac{16}{3}=\frac{36}{3}=12.$$

§15.3

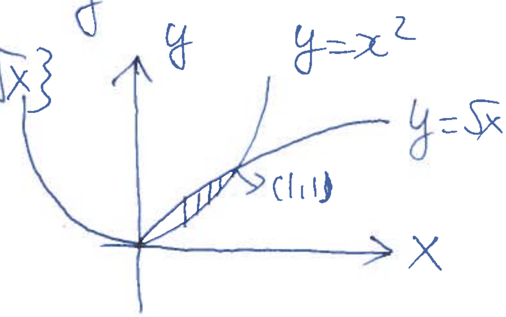
12. $\iint_D x\sqrt{y^2-x^2}\,dA \quad D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le y\}$

$$=\int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1 -\frac{1}{3}(y^2-x^2)^{\frac{3}{2}}\Big|_0^y dy$$

$$=\int_0^1 -\frac{1}{3}\left[0-y^3\right]dy = \int_0^1 \frac{y^3}{3}dy = \frac{y^4}{12}\Big|_0^1 = \frac{1}{12}$$

14. $\iint_D (x+y)\,dA$, D is bounded by $y=\sqrt{x}$ and $y=x^2$

$$\Rightarrow D=\{(x,y)\mid 0\le x\le 1,\ x^2\le y\le\sqrt{x}\}$$



$$=\int_0^1\int_{x^2}^{\sqrt{x}} x+y\,dy\,dx = \int_0^1 xy+\frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}} dx$$

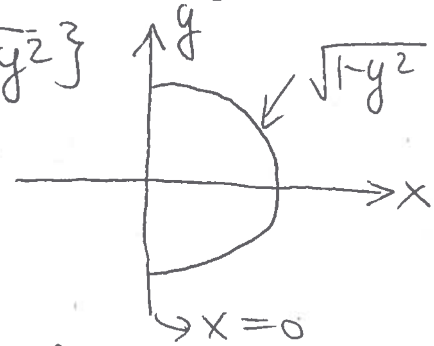
$$=\int_0^1 x\sqrt{x}+\frac{x}{2}-x^3-\frac{x^4}{2}\,dx = \frac{2}{5}x^{\frac{5}{2}}+\frac{x^2}{4}-\frac{x^4}{4}-\frac{x^5}{10}\Big|_0^1$$

$$=\frac{2}{5}+\frac{1}{4}-\frac{1}{4}-\frac{1}{10}=\frac{3}{10}$$

3 15.3

16. $\iint_D xy^2\,dA$, D is enclosed by $x=0$ and $x=\sqrt{1-y^2}$

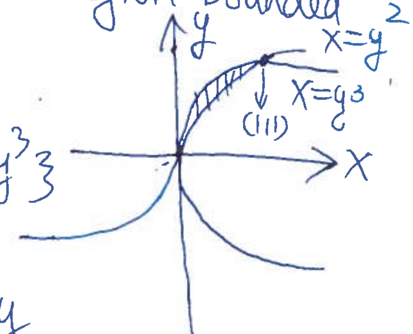
$\Rightarrow D=\{(x,y)\mid -1\le y\le 1,\ 0\le x\le \sqrt{1-y^2}\}$



$=\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy=\int_{-1}^{1}\frac{x^2y^2}{2}\Big|_0^{\sqrt{1-y^2}}dy$

$=\int_{-1}^{1}\frac{y^2(1-y^2)}{2}dy=\frac{y^3}{6}-\frac{y^5}{10}\Big|_{-1}^{1}=\frac{2}{6}-\frac{2}{10}=\frac{2}{15}$

20. The volume under $z=2x+y^2$ and above the region bounded by $x=y^2$ and $x=y^3$



The region $=\{(x,y)\mid 0\le y\le 1,\ y^2\le x\le y^3\}$

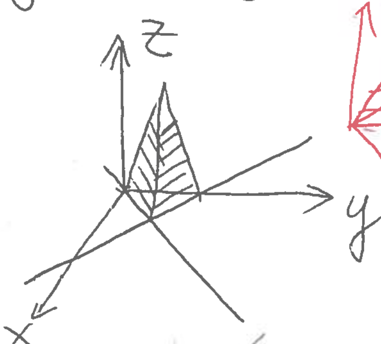
$\Rightarrow \int_0^1\int_{y^2}^{y^3} 2x+y^2\,dx\,dy=\int_0^1 x^2+xy^2\Big|_{y^2}^{y^3}dy$

$=\int_0^1 y^6+y^5-y^4-y^4\,dy=\frac{y^7}{7}+\frac{y^6}{6}-\frac{2y^5}{5}\Big|_0^1=\frac{1}{7}+\frac{1}{6}-\frac{2}{5}=\frac{30+35-84}{210}$

$=\frac{-19}{210}$

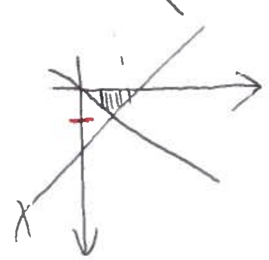
24. The volume bounded by $z=x$, $y=x$, $x+y=2$, $z=0$

$D=\{(x,y)\mid 0\le \cancel{y}\,x\le 1,\ \cancel{y}\,x\le \cancel{x}\,y\le 2-\cancel{y}\,x\}$



$\Rightarrow \iint_D x\,dx\,dy=\int_0^1\int_{\cancel{y}\,x}^{2-\cancel{y}\,x} x\,d\cancel{x}y\,d\cancel{y}x$

$=\int_0^1 \cancel{\frac{x^2}{2}}\,xy\Big|_{\cancel{y}\,x}^{2-\cancel{y}\,x} d\cancel{y}x \cancel{=\frac{1}{2}\int_0^1(2-y)^2-y^2dy}$



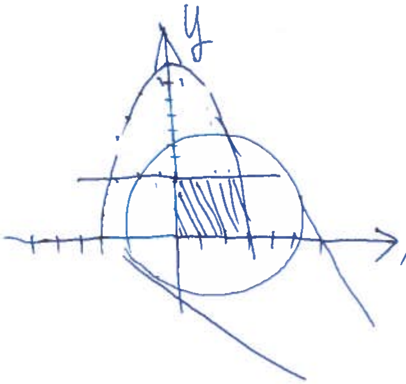
$\cancel{=\frac{1}{2}\int_0^1 4-4y\,dy=\frac{1}{2}(4y-2y^2)\Big|_0^1=1}$

$\int_0^1 x(2-2x)\,dx=x^2-\frac{2}{3}x^3\Big|_0^1=1-\frac{2}{3}=\frac{1}{3}$

42. $I=\int_0^3\int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy \Rightarrow D=\{(x,y)\,|\,0\le x\le\sqrt{9-y},\ 0\le y\le 3\}$

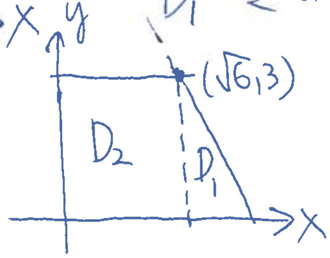
$\Downarrow$

$x^2=9-y \Rightarrow y=9-x^2$



changing the order $\Rightarrow$

$D_1=\{(x,y)\,|\,\sqrt{6}\le x\le 3,\ 0\le y\le 9-x^2\}\cup$



$D_2=\{(x,y)\,|\,0\le x\le\sqrt{6},\ 0\le y\le 3\}$

$$I=\int_0^3\int_0^{\sqrt{6}} f(x,y)\,dx\,dy+\int_{\sqrt{6}}^3\int_0^{9-x^2} f(x,y)\,dy\,dx$$

48. $\int_0^1\int_x^1 e^{\frac{x}{y}}\,dy\,dx \qquad D=\{(x,y)\,|\,0\le x\le 1,\ x\le y\le 1\}$

$\Rightarrow D=\{(x,y)\,|\,0\le y\le 1,\ 0\le x\le y\}$

$$=\int_0^1\int_0^y e^{\frac{x}{y}}\,dx\,dy=\int_0^1 ye^{\frac{x}{y}}\Big|_0^y dy=\int_0^1 [ey-y]\,dy$$

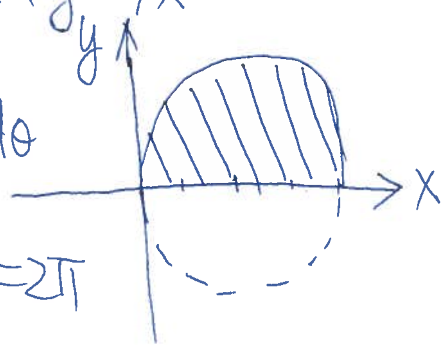
$$=(e-1)\frac{y^2}{2}\Big|_0^1=\frac{1}{2}(e-1)$$

§15.4

6. $\int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\,dr\,d\theta \Rightarrow 0\le\theta\le\frac{\pi}{2},\ 0\le r\le 4\cos\theta$

$\Rightarrow r=4\cos\theta \Rightarrow r^2=4r\cos\theta$

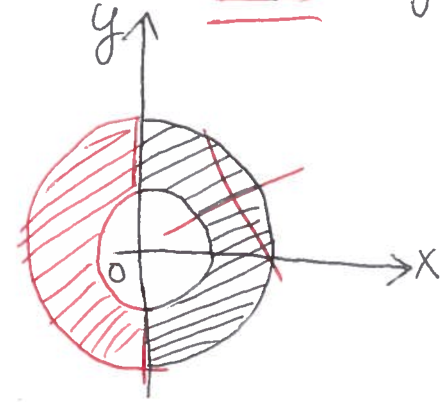
$=\int_0^{\frac{\pi}{2}}\frac{r^2}{2}\Big|_0^{4\cos\theta}d\theta \qquad \Rightarrow x^2+y^2=4x \Rightarrow (x-2)^2+y^2=4$



$=\int_0^{\frac{\pi}{2}} 8\cos^2\theta\,d\theta=8\int_0^{\frac{\pi}{2}}\frac{1-\cos2\theta}{2}\,d\theta$

$=4\left(\theta-\frac{1}{2}\sin2\theta\right)\Big|_0^{\frac{\pi}{2}}=4\cdot\frac{\pi}{2}=2\pi$

8. $\iint_R x+y\,dA$, where R is the region lies to the left of y-axis



between $x^2+y^2=1$ and $x^2+y^2=4$

$R=\{(r,\theta)\mid 1\le r\le 2,\ +\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}\}$

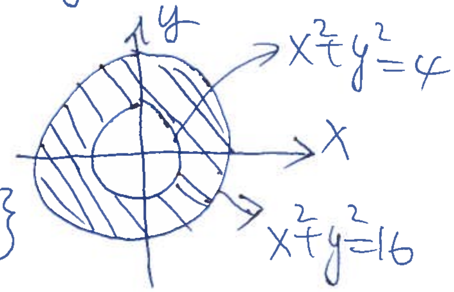
$$=\int_{+\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_1^2 (r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

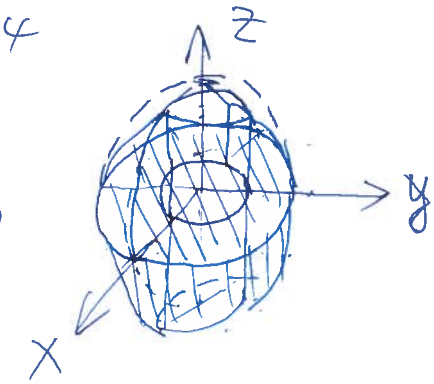
$$=\int_1^2 r^2\,dr\int_{+\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\theta+\sin\theta\,d\theta=\left[\frac{r^3}{3}\Big|_1^2\right]\cdot\Big[\sin\theta-\cos\theta\Big]_{+\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$=\cancel{\frac{7}{3}\cdot(1-0-(-1)-0)=\frac{14}{3}}\quad=\frac{7}{3}\left[-1-0-1+0\right]=-\frac{14}{3}$$

22. The volume inside $x^2+y^2+z^2=16$ and outside $x^2+y^2=4$

region for integration:





$R=\{(r,\theta)\mid 2\le r\le 4,\ 0\le\theta\le 2\pi\}$

the function we are integrating with

$z=\pm\sqrt{16-x^2-y^2}$

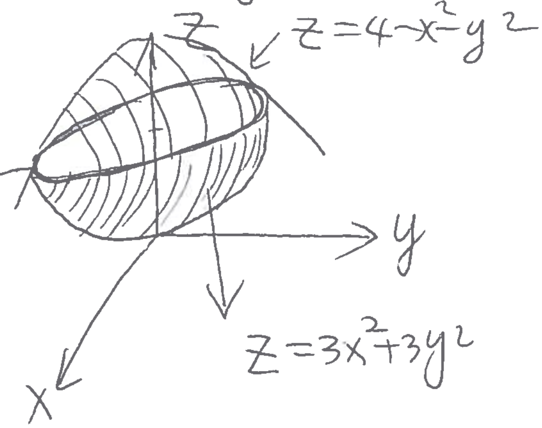
$$V=\iint_R \sqrt{16-x^2-y^2}-\left(-\sqrt{16-x^2-y^2}\right)dx\,dy$$

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $=2\int_2^4\int_0^{2\pi}\sqrt{16-r^2}\,r\,d\theta\,dr=2\cdot 2\pi\cdot\left[-\frac{1}{3}(16-r^2)^{\frac{3}{2}}\right]_2^4$

$$=4\pi\,\frac{1}{\cancel{3}}\cdot\overset{4}{\cancel{12}}\cdot 2\sqrt{3}=32\sqrt{3}\,\pi.$$

26. The volume between $z=3x^2+3y^2$ and $z=4-x^2-y^2$

The region D has a boundary



$$4-x^2-y^2=z=3x^2+3y^2$$

$$\Rightarrow 4=4x^2+4y^2 \Rightarrow x^2+y^2=1$$

The function we are integral with

$$4-x^2-y^2-(3x^2+3y^2)=4-4x^2-4y^2$$

$$V=\iint_D 4-4x^2-4y^2\,dxdy=\int_0^{2\pi}\int_0^1 (4-4r^2)r\,dr\,d\theta$$

$$=2\pi\int_0^1 4r-4r^3\,dr$$

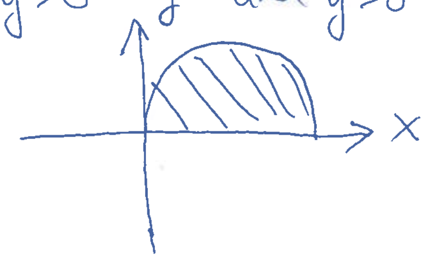
$$=8\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1=8\pi\cdot\frac{1}{4}=2\pi$$

32. $\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx$

The region $R=\{(x,y)\mid 0\le y\le\sqrt{2x-x^2},\ 0\le x\le 2\}$

$\Rightarrow y^2=2x-x^2 \Rightarrow (x-1)^2+y^2=1$ and $y>0$

and $y>0$



$x=r\cos\theta$
$y=r\sin\theta$

$$r^2\sin^2\theta=2r\cos\theta-r^2\cos^2\theta$$

$$\Rightarrow r^2(\sin^2\theta+\cos^2\theta)=2r\cos\theta$$

$$\Rightarrow r^2=2r\cos\theta \Rightarrow r=2\cos\theta$$

$$\Rightarrow 0\le r\le 2\cos\theta,\ 0\le\theta\le\frac{\pi}{2}$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{r^3}{3}\Big|_0^{2\cos\theta}d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos\theta(1-\sin^2\theta)\,d\theta$$

$$=\frac{8}{3}\left[\sin\theta-\frac{\sin^3\theta}{3}\Big|_0^{\frac{\pi}{2}}\right]$$

$$=\frac{8}{3}\left[1-\frac{1}{3}\right]=\frac{16}{9}$$

36. (a)

$$I=\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dA.$$ where D_a is disk with radius a and center O.

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta = \lim_{a\to\infty} 2\pi\int_0^a re^{-r^2}dr$$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta\end{cases}$

$$= \lim_{a\to\infty} 2\pi\left[\frac{-1}{2}e^{-r^2}\right]_0^a$$

$$= \lim_{a\to\infty} 2\pi\left[-\frac{1}{2}e^{-a^2}+\frac{1}{2}\right] = 2\pi\cdot\frac{1}{2}=\pi.$$

(b) $$\iint_{S_a} e^{-(x^2+y^2)}dA = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}dxdy = \left[\int_{-a}^{a} e^{-x^2}dx\right]\left[\int_{-a}^{a} e^{-y^2}dy\right]$$

So, by (a), we have

$$\pi=\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dA = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}dA = \lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}dx\right]\left[\int_{-a}^{a} e^{-y^2}dy\right]$$

$$=\int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy.$$

(c) by (b), let $y=x$, we have.

$$\pi=\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right] = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2.$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$$

(d) $$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx \overset{\boxed{\text{let } x=\sqrt{2}t,\ dx=\sqrt{2}dt}}{=\!=} \int_{-\infty}^{\infty} e^{-t^2}\cdot\sqrt{2}dt = \sqrt{2}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$=\sqrt{2\pi}.$$

(3)

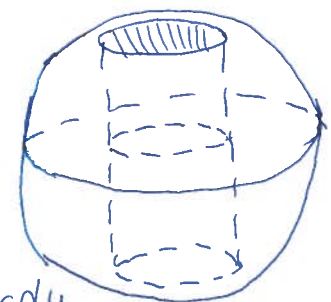
(a) Region of disc: $\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

Mass of disc $= \int_0^{2\pi}\int_0^1 r\,\rho(r,\theta)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(2-r)\,dr\,d\theta$

$= \cancel{2\pi \cdot \left[2r - \frac{r^2}{2}\right]_0^1 = 2\pi \cdot \frac{3}{2} = 3\pi}$ $= 2\pi \cdot \left[r^2 - \frac{r^3}{3}\right]_0^1 = \frac{4}{3}\pi$

(b) Region $D = \{(r,\theta) \mid 0 \le r \le \cancel{2}\,1,\ 0 \le \theta \le 2\pi\}$

function: $z = \pm\sqrt{4-x^2-y^2}$



$$V = \iint_D \sqrt{4-x^2-y^2} - \left(-\sqrt{4-x^2-y^2}\right)dx\,dy$$

$$= 2\int_0^1\int_0^{2\pi} \sqrt{4-r^2}\ r\,d\theta\,dr = 2\cdot 2\pi\left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right]_0^1$$

$\cancel{= 4\pi \cdot \left[\frac{1}{3} 3\sqrt{3}\right] = 4\sqrt{3}\pi}$ $\quad 4\pi\left[\frac{1}{3}(8-3\sqrt{3})\right]$

$$= \frac{4(8-3\sqrt{3})}{3}\pi$$