Honors Calculus, Math 1451 - HW5 Solutions

§14.6

6, 8, 12, 37, See HW4 Solutions

28. Given $f(x,y) = ye^{-xy}$ and a point $(0,2)$.

To find the directions $\vec{u} = \langle a, b \rangle$ such that

$$D_{\vec{u}} f(x,y)\Big|_{(0,2)} = 1,$$

We have

$$D_{\vec{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$
$$= \langle -y^2 e^{-xy}, e^{-xy} - xye^{-xy} \rangle \cdot \langle a, b \rangle$$

and $D_{\vec{u}} f(0,2) = \langle -4, 1 \rangle \cdot \langle a, b \rangle = -4a + b = 1$.

Since $\vec{u}$ is an unit vector, we have $a^2 + b^2 = 1$.

$$\Rightarrow a^2 + (1+4a)^2 = 1 \Rightarrow a^2 + 16a^2 + 8a + 1 = 1 \Rightarrow a(17a+8) = 0$$

$$\Rightarrow a = 0 \text{ or } a = -\frac{8}{17} \Rightarrow b = 1 \text{ or } b = \frac{-15}{17}$$

$$\Rightarrow \vec{u} = \langle 0, 1 \rangle \text{ or } \langle -\frac{8}{17}, -\frac{15}{17} \rangle.$$

51. Given an elliptic paraboloid $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and a point (x_0, y_0, z_0);

To Find the tangent plane of this elliptic paraboloid at the given point, we have

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}$$

and $Df(x,y,z)=\langle f_x, f_y, f_z\rangle=\langle \frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{1}{c}\rangle$

Then the normal vector of the tangent plane is

$$Df(x_0,y_0,z_0)=\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, -\frac{1}{c}\rangle$$

Thus the tangent plane equation is

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)-\frac{1}{c}(z-z_0)=0.$$

$$\Rightarrow \frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\underbrace{\frac{2x_0^2}{a^2}+\frac{2y_0^2}{b^2}}_{=\frac{2z_0}{c}}+\frac{z-z_0}{c}$$

$$\Rightarrow \frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z+z_0}{c}$$

§14.7

6. Given $f(x,y)=x^3y+12x^2-8y$. Since f is a polynomial, $f\in C^\infty$.

To find local max, min, and saddle points, first,

we need to find critical points (a,b) such that $f_x(a,b)=0$, $f_y(a,b)=0$.

$\Rightarrow f_x(x,y)=3x^2y-24x=0$, $f_y(x,y)=x^3-8=0$ $\Rightarrow x=2$ & $3x^2y-24x=0$

$\Rightarrow x=2, y=4$ $\Rightarrow (2,4)$ is the only critical point.

Second, using second derivatives test, we have

$f_{xx}(x,y)=6xy-24$, $f_{xy}(x,y)=3x^2$, $f_{yy}(x,y)=0$

$D(x,y)=f_{xx}(x,y)f_{yy}(x,y)-[f_{xy}(x,y)]^2=-[3x^2]^2$

Since $D(2,4)<0$ and by second derivatives test (c), we have

$(2,4)$ is a saddle point.

12. Given $f(x,y)=xy+\frac{1}{x}+\frac{1}{y}$. f is NOT continuous at $(0,0)$ or on $x=0$ or on $y=0$.

To find local max, min, and saddle points.

We need to find critical point (a,b) such that $f_x(a,b)=0$, $f_y(a,b)=0$.

$\Rightarrow f_x(x,y)=y-\frac{1}{x^2}=0$, $f_y(x,y)=x-\frac{1}{y^2}=0$

$\Rightarrow y=\frac{1}{x^2}$ and $x=\frac{1}{y^2}$ $\Rightarrow x=x^4$ $\Rightarrow x(x^3-1)=0$ $\Rightarrow x=0$ or 1

(but x cannot be 0)

$\Rightarrow x=1, y=1$ $\Rightarrow (1,1)$ is a critical point.

Second, using second derivatives test, we have

$f_{xx}(x,y)=\frac{2}{x^3}$, $f_{xy}(x,y)=1$, $f_{yy}(x,y)=\frac{2}{y^3}$

$D(x,y) = \frac{2}{x^3} \cdot \frac{2}{y^3} - 1^2 = \frac{4}{x^3y^3} - 1.$

since $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$

$\Rightarrow$ by Second derivatives Test (a), (1,1) is a local min.

16. Given $f(x,y) = e^y(y^2 - x^2)$. f is ~~an infinity~~ a smooth function.
To find local max, min., and saddle points, we need to find the critical point (a,b) such that $f_x(a,b) = 0$, $f_y(a,b) = 0$

$\Rightarrow f_x(x,y) = -2xe^y = 0, \quad f_y(x,y) = e^y(y^2 - x^2) + 2ye^y = 0$

$\Rightarrow x = 0, \ e^y(y^2 + 2y) = 0 \Rightarrow x = 0, \ y = 0$ or -2

(0,0) and (0,-2) are critical points.

Then, using Second derivatives test, we need.

$f_{xx}(x,y) = -2e^y, \quad f_{xy}(x,y) = -2xe^y, \quad f_{yy}(x,y) =$
$e^y(y^2 + 2y - x^2) + e^y(2y + 2).$

Thus,

① For (0,0), we have

$D(0,0) = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -2 \cdot 2 - 0 < 0.$

and, by Second derivatives Test (c), (0,0) is a saddle point.

② For (0,-2),

$D(0,-2) = (-2e^{-2}) \cdot (-2e^{-2}) - 0 = 4e^{-2} > 0$ and

$f_{xx}(0,-2) = -2e^{-2} < 0 \Rightarrow (0,-2)$ is a local max.

23. Given $f(x,y) = \sin(x) + \sin(y) + \sin(x+y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

Critical points $\Rightarrow$ $f_x(x,y) = \cos(x) + \cos(x+y)$, $f_y(x,y) = \cos(y) + \cos(x+y)$

$f_x = 0$, $f_y = 0$ $\Rightarrow$ $\cos(x) = \cos(y)$ $\Rightarrow$ $x = y$ or $x = 2\pi - y$.

If $x = y$, $f_x = \cos(x) + \cos(2x) = 0 \Rightarrow \cos(x) + 2\cos^2(x) - 1 = 0$

$\Rightarrow \cos(x) = -1$ or $\frac{1}{2}$ $\Rightarrow$ $x = \pi, \frac{\pi}{3}$ or $\frac{5\pi}{3}$, we have (π, π), $(\frac{\pi}{3}, \frac{\pi}{3})$ $(\frac{5\pi}{3}, \frac{5\pi}{3})$

If $x = 2\pi - y$, $f_x = \cos(x) + 1 = 0 \Rightarrow x = \pi$, we have (π, π).

Second Derivatives Test: $f_{xx} = -\sin(x) - \sin(x+y)$, $f_{xy} = -\sin(x+y)$

$f_{yy} = -\sin(y) - \sin(x+y)$

$D(x,y) = \sin(x)\sin(y) + \sin(y)\sin(x+y) + \sin(x)\sin(x+y)$.

① $D(\frac{\pi}{3}, \frac{\pi}{3}) = (\frac{\sqrt{3}}{2})^2 + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} > 0$. $f_{xx}(\frac{\pi}{3}, \frac{\pi}{3}) < 0 \Rightarrow$ local max.

② $D(\frac{5\pi}{3}, \frac{5\pi}{3}) = (\frac{-\sqrt{3}}{2})^2 + (-\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) + (-\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) > 0$. $f_{xx}(\frac{5\pi}{3}, \frac{5\pi}{3}) > 0 \Rightarrow$ local min.

③ $D(\pi, \pi) = 0$, Second derivatives test fails, but as $x = y$,

we have $f(x,x) = 2\sin(x) + \sin(2x) = 2\sin(x)(1 + \cos(x))$

$f(x,x) > 0$ as $0 < x < \pi$, $f(x,x) < 0$ as $\pi < x < 2\pi$

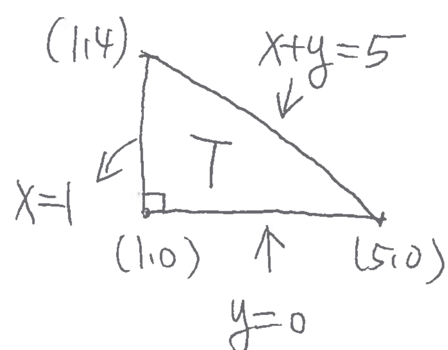
$\Rightarrow$ In a small disc of (π, π), some has positive f, some are negative

$\Rightarrow$ saddle point!

30. Given $f(x,y) = 3 + xy - x - 2y$ and T = closed triangular region with Vertices $(1,0)$, $(5,0)$ $(1,4)$

(I) Check the critical points of f in T.

$f_x(x,y) = y - 1 = 0$, $f_y(x,y) = x - 2 = 0$



$\Rightarrow (2,1)$ is a critical point of f and It is in region T.

Using Second derivatives Test, we have

$f_{xx}(x,y) = 0$, $f_{xy}(x,y) = 1$. $f_{yy}(x,y) = 0 \Rightarrow D(2,1) < 0$

$\Rightarrow (2,1)$ is a saddle point (NOT local min or max).

(II) Check the point ~~at~~ of the boundary of T:

As $x=1$, we have $f(1,y)=3+y-1-2y$, $0\le y\le 4$.
$=2-y$

Since $f(1,y)$ is a decreasing function.

$\Rightarrow$ local max. at $(1,0)$ and $f(1,0)=2$.
local min. at $(1,4)$ and $f(1,4)=-2$

As $y=0$, we have $f(x,0)=3-x$, $1\le x\le 5$.

Since $f(x,0)$ is a decreasing function.

$\Rightarrow$ local max. at $(1,0)$ and $f(1,0)=2$.
local min. at $(5,0)$ and $f(5,0)=-2$

As $x+y=5 \Rightarrow y=5-x$, $1\le x\le 5$, we have.

$$f(x,y)=f(x,5-x)=3+x(5-x)-x-2(5-x), \quad 1\le x\le 5$$
$$=3+5x-x^2-x-10+2x$$
$$=-x^2+6x-7$$

and $\left.\frac{\partial f}{\partial x}\right|_{x+y=5}=\frac{df}{dx}=-2x+6=0$, $x=3$ and $\frac{d^2f}{dx^2}=-2<0$

$\Rightarrow$ local max at $(3,2)$ and $f(3,2)=2$.
local min at $(1,4)$ and $f(1,4)=-2$
local min at $(5,0)$ and $f(5,0)=-2$

By (I) & (II)

$\Rightarrow$ Absolute max of f on T is 2.

Absolute min of f on T is -2.

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§14.7

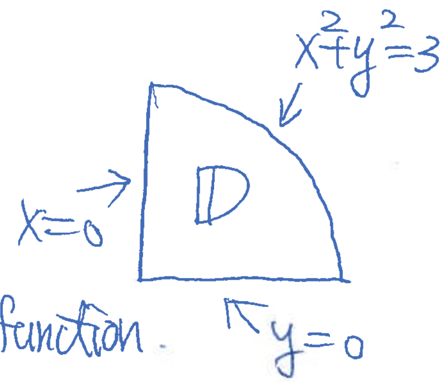
34. Given $f(x,y)=xy^2$, $D=\{(x,y)\mid x\geq 0,\ y\geq 0,\ x^2+y^2\leq 3\}$

(I) To find the critical points in D, we have.

$f_x(x,y)=y^2=0$, $f_y(x,y)=2xy=0$ $\Rightarrow$ $(0,0)$ is a critical point of f, but $(0,0)$ is a point of the boundary of D.

Let's go to case (II).

(II)



For the points of $x=0$, we have

$f(0,y)=0$, $0\leq y\leq\sqrt{3}$ $\Rightarrow$ constant function.

For the points of $y=0$, we have $f(x,0)=0$, $0\leq x\leq\sqrt{3}$ $\Rightarrow$ constant

For the points of $x^2+y^2=3$ $\Rightarrow$ $y^2=3-x^2$, $0\leq x\leq\sqrt{3}$

$f(x,y)=f(x,\sqrt{3-x^2})=x(3-x^2)=3x-x^3$, $0\leq x\leq\sqrt{3}$.

$\left.\frac{df}{dx}\right|_{x^2+y^2=3}=3-3x^2=0 \Rightarrow x=1$ or $\cancel{-1}$, $\left.\frac{d^2f}{dx^2}\right|_{x^2+y^2=3}=-6x<0$ as $x=1$

$\Rightarrow$ as $x=1$, $y=\sqrt{2}$, $f(1,\sqrt{2})=2$ is a local max.

as $x=0$, $y=\sqrt{3}$, $f(0,\sqrt{3})=0$ is a local min

as $x=\sqrt{3}$, $y=0$, $f(\sqrt{3},0)=0$ is a local min.

By (I) & (2)

$\Rightarrow$ Absolute max of $f(x,y)$ on D is 2.

Absolute min of $f(x,y)$ on D is 0.

40. Given plane $x-y+z=4$ and a point $(1,2,3)$.

To find the closest point on Given plane to given point,

We have

(Method 1). Using the method of 14.7. we need to find the point (x,y,z). such that

$f(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$ has an absolute min. as

$$x-y+z=4. \Rightarrow z=4-x+y.$$

Then $f(x,y,z)=f(x,y,4-x+y)=(x-1)^2+(y-2)^2+(4-x+y-3)^2$

$$=(x-1)^2+(y-2)^2+(1-x+y)^2$$

By Second Derivatives Test, we need to find the critical points.

$f_x(x,y,z)=2(x-1)-2(1-x+y)=4x-2y-4=0$ and.

$f_y(x,y,z)=2(y-2)+2(1-x+y)=-2x+4y-2=0$

$\Rightarrow \begin{cases} 4x-2y=4 \\ 2x-4y=-2 \end{cases} \Rightarrow x=\frac{5}{3},\ y=\frac{4}{3} \Rightarrow (\frac{5}{3},\frac{4}{3})$ is a critical point

Then $f_{xx}(x,y,z)=4$. $f_{xy}(x,y,z)=-2$. $f_{yy}(x,y,z)=4$.

$D(x,y,z)=16-(-2)^2>0$ and $f_{xx}>0$.

So as $x=\frac{5}{3}$, $y=\frac{4}{3}$. $f(x,y,z)$ has min.

Thus the point is $\underline{(\frac{5}{3},\frac{4}{3},\frac{11}{3})}$ $(z=4-x+y)$.

(Method 2). We can find a parametric equation of line which is perpendicular to the plane and pass the given point. Then it is easy to find out the point we want:

The line should have the same direction of the normal vector of the plane $\langle 1,-1,1\rangle$, so the parametric equation of this line is $x=t+1,\ y=-t+2,\ z=t+3$.

Putting this parameter back to the plane, we have

$$(t+1)-(-t+2)+(t+3)=4 \Rightarrow 3t=4-2 \Rightarrow t=\frac{2}{3}.$$

So $x=\frac{2}{3}+1=\frac{5}{3},\ y=-\frac{2}{3}+2=\frac{4}{3},\ z=\frac{2}{3}+3=\frac{11}{3}$.

$\Rightarrow$ the point is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

54. Given $P(p,q,r)=2pq+2pr+2rq$ and $p+q+r=1$.

To show P is at most $\frac{2}{3}$ $\Rightarrow$ find the max. of P as $p+q+r=1$.

Since $p+q+r=1 \Rightarrow r=1-p-q$, we have

$$P(p,q,r)=P(p,q,1-p-q)=2pq+2(p+q)(1-p-q)$$

Then, to find the critical points of $P(p,q,1-p-q)$, we have

$$\frac{\partial P}{\partial p}=2q+2(1-p-q)+2(p+q)(-1)=2-4p-2q=0$$

$$\frac{\partial P}{\partial q}=2p+2(1-p-q)+2(p+q)(-1)=2-2p-4q=0$$

$$\Rightarrow \begin{cases} 4p+2q=2 \\ 2p+4q=2 \end{cases} \Rightarrow p=\frac{1}{3},\ q=\frac{1}{3}$$

$$\underset{<0}{\frac{\partial^2 P}{\partial p^2}=-4},\quad \frac{\partial^2 P}{\partial p\partial q}=-2,\quad \frac{\partial^2 P}{\partial q^2}=-4.\quad D=\frac{\partial^2 P}{\partial p^2}\cdot\frac{\partial^2 P}{\partial q^2}-\left(\frac{\partial^2 P}{\partial p\partial q}\right)^2=16-(-2)^2>0$$

$\Rightarrow$ as $p=\frac{1}{3},\ q=\frac{1}{3}$, P has a local max.

which is $P(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 2\cdot\frac{1}{3}\cdot\frac{1}{3} + 2\cdot\frac{1}{3}\cdot\frac{1}{3} + 2\cdot\frac{1}{3}\cdot\frac{1}{3}$

$$= \frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}.$$

§14.8

6. Given $f(x,y) = e^{xy}$ and a given constraint $x^3+y^3=16$.

Let $g(x,y) = x^3+y^3=16$. Then $\nabla f = \langle ye^{xy}, xe^{xy}\rangle$, $\nabla g = \langle 3x^2, 3y^2\rangle$

Using Lagrange Multipliers, we solve the equations

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^3+y^3=16 \end{cases} \Rightarrow \begin{cases} ye^{xy} = \lambda 3x^2 & (1) \\ xe^{xy} = \lambda 3y^2 & (2) \\ x^3+y^3=16 & (3) \end{cases}$$

$\left(\begin{array}{l} \text{if } x=0 \Rightarrow y=0 \\ \text{but } x^3+y^3=16 \neq 0 \\ \Rightarrow x\neq 0,\ y\neq 0 \end{array}\right)$

From (1) & (2), we have $\frac{3x^2}{y} = \frac{e^{xy}}{\lambda} = \frac{3y^2}{x} \Rightarrow x^3 = y^3 \Rightarrow x = y$

As $x=y$, $\xRightarrow{\text{by (3)}} 2x^3=16 \Rightarrow x^3=8 \Rightarrow x=2, \Rightarrow (2,2)$.

Since we can choose (x,y) satisfied $x^3+y^3=16$ such that $f(x,y)$ is very closed to zero, but not zero, so $f(2,2) = e^4$ is a maximum.

8. Given $f(x,y,z) = 8x - 4z$ and constraint $x^2 + 10y^2 + z^2 = 5$.

Let $g(x,y,z) = x^2 + 10y^2 + z^2 = 5$.

Then $\nabla f = \langle 8, 0, -4 \rangle$, $\nabla g = \langle 2x, 20y, 2z \rangle$.

By Lagrange Multipliers, we solve the equations.

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + 10y^2 + z^2 = 5 \end{cases} \Rightarrow \begin{cases} 8 = 2\lambda x & \text{—(1)} \\ 0 = 20\lambda y & \text{—(2)} \\ -4 = 2\lambda z & \text{—(3)} \\ x^2 + 10y^2 + z^2 = 5 & \text{—(4)} \end{cases}$$

From (2) $\Rightarrow \lambda = 0$ or $y = 0$, but, by (1), (3), $\lambda \neq 0$

$\Rightarrow y = 0$. $\Rightarrow 2\lambda x = -2 \cdot 2\lambda z \Rightarrow x = -2z$.

By (4) $\Rightarrow (-2z)^2 + 0 + z^2 = 5 \Rightarrow z^2 = 1$, $z = \pm 1$, $x = \mp 2$

$\Rightarrow$ two points $(-2, 0, 1)$ and $(2, 0, -1)$.

Since $f(-2,0,1) = -16 - 4 = -20$ and

$f(2,0,-1) = 16 + 4 = 20$,

Then, under the constraint, f has maximum 20 at $(2,0,-1)$

and f has minimum -20 at $(-2,0,1)$.

10. Given $f(x,y,z) = x^2y^2z^2$, and constraint $x^2+y^2+z^2=1$.

Let $g(x,y,z) = x^2+y^2+z^2=1$, we have

$\nabla f = \langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z \rangle$, $\nabla g = \langle 2x, 2y, 2z \rangle$.

By Lagrange Multipliers, we solve the equations:

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2+z^2=1 \end{cases} \Rightarrow \begin{cases} 2xy^2z^2 = 2\lambda x \\ 2x^2yz^2 = 2\lambda y \\ 2x^2y^2z = 2\lambda z \\ x^2+y^2+z^2=1 \end{cases} \xRightarrow[\substack{y\neq 0 \\ z\neq 0}]{x\neq 0} \begin{cases} 2y^2z^2 = 2\lambda \quad \text{—(1)} \\ 2x^2z^2 = 2\lambda \quad \text{—(2)} \\ 2x^2y^2 = 2\lambda \quad \text{—(3)} \\ x^2+y^2+z^2=1 \quad \text{—(4)} \end{cases}$$

$$\left(\begin{array}{l} \text{If } x=0, \Rightarrow y=0, z=0 \\ \text{but } x^2+y^2+z^2=1\neq 0 \end{array} \right) \Rightarrow \text{by (1), (2) \& (3),} \quad y^2z^2=x^2z^2,\ x^2z^2=x^2y^2,\ y^2z^2=x^2y^2$$

$$\Rightarrow x^2=y^2,\ x^2=y^2,\ z^2=x^2$$

$\Rightarrow x^2=y^2=z^2$ since $x^2+y^2+z^2=1 \Rightarrow x^2=y^2=z^2=\frac{1}{3}$

We have eight points $\left(\pm\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(\pm\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(\pm\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{-1}{\sqrt3}\right)$, $\left(\pm\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}\right)$

Furthermore, as $\lambda=0$, we have one or two of x, y, z are 0, for example $x=0, y=0, z=1$ which gets a min. value of $f(0,0,1)=0$

So $f\left(\pm\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = f\left(\pm\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right) = f\left(\pm\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right) = f\left(\pm\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}\right)$

$= \frac{1}{27}$ is a max. under the constraint.

§14.8

14. Given $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ and constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Let $g(x_1, x_2, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

Then $\nabla f = (1, 1, 1, 1, \cdots, 1)$ and $\nabla g = (2x_1, 2x_2, \cdots, 2x_n)$

By Lagrange Multipliers, we solve the equation.

$$\begin{cases} \nabla f = \lambda \nabla g \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases} \Rightarrow \begin{cases} 1 = 2\lambda x_1 & -(1) \\ 1 = 2\lambda x_2 & -(2) \\ \vdots & \vdots \\ 1 = 2\lambda x_n & -(n) \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 & -(n+1) \end{cases}$$

From (1) – (n), we have $x_1 = x_2 = x_3 = \cdots = x_n$.

Then, by (n+1), we obtain $x_1^2 = x_2^2 = x_3^2 = \cdots = x_n^2 = \frac{1}{n}$.

$\Rightarrow x_i = \pm \frac{1}{\sqrt{n}}$ for $i = 1, \cdots, n$

So, under the constraint,

at $\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right)$ f has max. value $\frac{n}{\sqrt{n}} = \sqrt{n}$.

and at $\left(\frac{-1}{\sqrt{n}}, \frac{-1}{\sqrt{n}}, \cdots, \frac{-1}{\sqrt{n}}\right)$, f has min. value $\frac{-n}{\sqrt{n}} = -\sqrt{n}$.

46. Given $f(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} x_i y_i$ and two constraints.

(a)

$g(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} x_i^{2} = 1$ and

$h(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} y_i^2 = 1$. Then we have

$\nabla f = \langle y_1, y_2, y_3, \dots, y_n, x_1, x_2, x_3, \dots, x_n \rangle$.

$\nabla g = \langle 2x_1, 2x_2, 2x_3, \dots, 2x_n, 0, 0, 0, \dots, 0 \rangle$

$\nabla h = \langle 0, 0, 0, 0, \dots, 0, 2y_1, 2y_2, 2y_3, \dots, 2y_n \rangle$

By Lagrange Multipliers, we solve the equations

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(,\dots) = 1 \\ h(,\dots) = 1 \end{cases} \Rightarrow \begin{cases} y_1 = 2\lambda x_1 + 0 & -(1) \\ y_2 = 2\lambda x_2 + 0 & -(2) \\ y_3 = 2\lambda x_3 + 0 & -(3) \\ \vdots \\ y_n = 2\lambda x_n + 0 & -(n) \\ x_1 = 0 + 2\mu y_1 & -(n+1) \\ x_2 = 0 + 2\mu y_2 & -(n+2) \\ \vdots & \vdots \\ x_n = 0 + 2\mu y_n & -(2n) \\ x_1^2 + \dots + x_n^2 = 1 & -(2n+1) \\ y_1^2 + \dots + y_n^2 = 1 & -(2n+2) \end{cases} \Rightarrow \begin{matrix} y_1 = 4\lambda\mu y_1 \\ y_2 = 4\lambda\mu y_2 \\ \vdots \\ y_n = 4\lambda\mu y_n \end{matrix} \Rightarrow \lambda\mu = \frac{1}{4}$$

Using (1)–(n) and (2n+2), we have

$(2\lambda)^2 \left[x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 \right] = 1$.

But, by (2n+1), $x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = 1 \Rightarrow (2\lambda)^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2}$.

$\Rightarrow \mu = \pm \frac{1}{2}$

As $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, we have

$$y_1 = x_1$$
$$y_2 = x_2$$
$$\vdots$$
$$y_n = x_n$$

under the constranits,

f has maximum value

$$f = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2 = 1$$

As $\lambda = -\frac{1}{2}$, $\mu = -\frac{1}{2}$, we have

$$y_1 = -x_1$$
$$y_2 = -x_2$$
$$\vdots$$
$$y_n = -x_n$$

Under the constraints,

f has min. value

$$f = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} -x_i x_i = -\sum_{i=1}^{n} x_i^2 = -1$$

(b) If $x_i = \dfrac{a_i}{\sqrt{\Sigma a_j^2}}$, $y_i = \dfrac{b_i}{\sqrt{\Sigma b_j^2}}$, by part (a), we have

$$-1 \le \sum_{i=1}^{n} x_i y_i \le 1 \Rightarrow \sum_{i=1}^{n} \frac{a_i b_i}{\sqrt{\Sigma a_j^2}\sqrt{\Sigma b_j^2}} \le 1$$

$$\Rightarrow \sum_{i=1}^{n} a_i b_i \le \sqrt{\Sigma a_j^2}\sqrt{\Sigma b_j^2}$$

(since $\Sigma x_i^2 = \Sigma \dfrac{a_i^2}{\Sigma a_j^2} = \dfrac{\Sigma a_i^2}{\Sigma a_j^2} = 1$, and

$\Sigma y_i^2 = \Sigma \dfrac{b_i^2}{\Sigma b_j^2} = \dfrac{\Sigma b_i^2}{\Sigma b_j^2} = 1$. so we can use part (a))

(4) A costs 11 euros / unit. B costs 3 euros.

Given $f(x,y) = -3x^2 + 10xy - 3y^2$.

with x units A, y units B, $f(x,y)$ units C.

The cost function $C(x,y) = 11x + 3y$. and

~~the~~ constraint is $f(x,y) = -3x^2 + 10xy - 3y^2 = 80$

To Find min of C by Lagrange Multipliers,

We have $\nabla C(x,y) = \langle 11, 3 \rangle$, $\nabla f(x,y) = \langle -6x + 10y, 10x - 6y \rangle$

and $\begin{cases} \nabla C = \lambda \nabla f \\ f(x,y) = 80 \end{cases} \Rightarrow \begin{cases} 11 = \lambda(-6x + 10y) & -(1) \\ 3 = \lambda(10x - 6y) & -(2) \\ -3x^2 + 10xy - 3y^2 = 80 & -(3) \end{cases}$

$(\lambda \neq 0)$

From (1), (2), we have

$\begin{cases} -6x + 10y = \frac{11}{\lambda} \\ 10x - 6y = \frac{3}{\lambda} \end{cases} \Rightarrow x = \frac{48}{32\lambda} = \frac{3}{2\lambda}, \; y = \frac{2}{\lambda}$

Put this back to (3) $\Rightarrow -3 \cdot \left(\frac{3}{2\lambda}\right)^2 + 10 \frac{3}{\cancel{2}\lambda} \cdot \frac{\cancel{2}}{\lambda} - 3\left(\frac{2}{\lambda}\right)^2 = 80$

$\Rightarrow \frac{-27}{4\lambda^2} + \frac{30}{\lambda^2} - \frac{12}{\lambda^2} = 80 \Rightarrow \frac{45}{4\lambda^2} = 80 \Rightarrow \lambda^2 = \frac{45}{4 \cdot 80} = \frac{9}{4 \cdot 16}$

$\Rightarrow \lambda = \pm \frac{3}{8} \Rightarrow x = \pm 4, \; y = \pm \frac{16}{3} \quad (x > 0, y > 0)$

$\Rightarrow x = 4, \; y = \frac{16}{3} \Rightarrow C(4, \frac{16}{3}) = 44 + 16 = 60$ euros

will be the minimum.

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