Honors Calculus, Math 1451 − HW4 Solution

§14.4

12. Given $f(x,y)=x^3y^4$ and a point $(1,1)$.

- Differentiability: Since $f(x,y)$ is a polynomial which is also continuous on its own domain and

  $f_x(x,y)\big|_{(1,1)}=3x^2y^4\big|_{(1,1)}=3$, $f_y(x,y)\big|_{(1,1)}=4x^3y^3=4$ exist, Then

  $f(x,y)$ is differentiable at $(1,1)$

- Linearization: $\left(f(x,y)\approx f(1,1)+f_x(1,1)\cdot(x-1)+f_y(1,1)\cdot(y-1)\right)$

  $$f(x,y)\approx f(1,1)+3(x-1)+4(y-1)=1+3(x-1)+4(y-1)$$

14. Given $f(x,y)=\sqrt{x+e^{4y}}$ and a point $(3,0)$.

- Differentiability:

  Since $f(x,y)$ is continuous on its own domain $\{(x,y)\mid x+e^{4y}>0\}$ and $(3,0)$ is in this set, and

  $f_x(x,y)\big|_{(3,0)}=\frac{1}{2\sqrt{x+e^{4y}}}\Big|_{(3,0)}=\frac{1}{4}$, $f_y(x,y)\big|_{(3,0)}=\frac{4e^{4y}}{2\sqrt{x+e^{4y}}}=1$ exist.

  Then $f(x,y)$ is differentiable at $(3,0)$

- Linearization:

  $$f(x,y)\approx f(3,0)+f_x(3,0)(x-3)+f_y(3,0)(y-0)$$

  $$=2+\frac{1}{4}(x-3)+y$$

18. Given $f(x,y)=\sqrt{y+\cos^2(x)}$ and point $(0,0)$. Then the linear approximation of $f(x,y)$ at $(0,0)$ is

$$f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)$$

$$=\sqrt{0+\cos^2 0}+\left.\frac{-2\cos(x)\sin(x)}{2\sqrt{y+\cos^2(x)}}\right|_{(0,0)}\cdot x+\left.\frac{1}{2\sqrt{y+\cos^2(x)}}\right|_{(0,0)} y = 1+\frac{y}{2}$$

40. Given four positive numbers x,y,z,w, the product function f will be $f(x,y,z,w)=xyzw$. Then the differential of f is

$$df=f_x\cdot dx+f_y\,dy+f_z\,dz+f_w\,dw = yzw(\Delta x)+xzw(\Delta y)+xyw(\Delta z)+xyz(\Delta w)$$

$$\boxed{\text{let } dx=\Delta x \quad dz=\Delta z \\ dy=\Delta y \quad dw=\Delta w}$$

Since each of the number less than 50 $\Rightarrow |x|<50, |y|<50, |z|<50, |w|<50$

and we do first decimal place rounding $\Rightarrow |\Delta x|<0.05, |\Delta y|<0.05, |\Delta z|<0.05, |\Delta w|<0.05$.

Then $|df| < 50^3\cdot 0.05+50^3\cdot 0.05+50^3\cdot 0.05+50^3\cdot 0.05 = 25000$.

42. Given two curves on the surface S:

$\vec{r}_1(t)=\langle 2+3t, 1-t^2, 3-4t+t^2\rangle$ and $\vec{r}_2(u)=\langle 1+u^2, 2u^3-1, 2u+1\rangle$

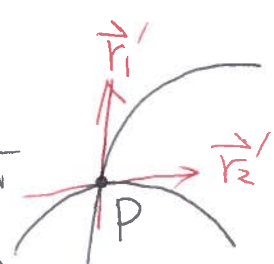
and a point $P(2,1,3)$.

For the tangent plane of S at P. We have two vectors on the tangent plane $\vec{r}_1{}'(t)=\langle 3,-2t,-4+2t\rangle$ and

$\vec{r}_2{}'(u)=\langle 2u, 6u^2, 2\rangle$.

Since we have to find t_0 and u_0 such that

$\vec{r}_1(t_0)=\vec{r}_2(u_0)=\langle 2,1,3\rangle$ and $\vec{r}_1{}'(t_0), \vec{r}_2{}'(u_0)$



are two vectors on the tangent plane. So

$t_0 = 0$ and $u_0 = 1$ $\left(\begin{cases} 2+3t_0 = 2 \\ 1-t_0^2 = 1 \\ 3-4t_0+t_0^2 = 3 \end{cases} \Rightarrow t_0 = 0 \text{ and } \begin{cases} 1+u_0^2 = 2 \\ 2u_0^3 - 1 = 1 \\ 2u_0+1 = 3 \end{cases} \Rightarrow u_0 = 1 \right)$

and $\vec{r}_1{}'(t_0) = \langle 3, 0, -4 \rangle$, $\vec{r}_2{}'(u_0) = \langle 2, 6, 2 \rangle$. Then the normal vector of this plane is $\vec{n} = \vec{r}_1{}'(t_0) \times \vec{r}_2{}'(u_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix}$

$$= 24\vec{i} - 14\vec{j} + 18\vec{k}$$
$$= 2\langle 12, -7, 9 \rangle$$

So the equation of the tangent plane at point p is

$$12x - 7y + 9z = 44.$$

§14.5

6. Given $w = \ln\sqrt{x^2+y^2+z^2}$ $\left(\underset{\shortparallel}{\tfrac{1}{2}\ln(x^2+y^2+z^2)}\right)$ and $x = \sin(t)$, $y = \cos(t)$, $z = \tan(t)$

Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\cancel{2}x}{\cancel{2}(x^2+y^2+z^2)} \cdot \cos(t) + \frac{\cancel{2}y}{\cancel{2}(x^2+y^2+z^2)} \cdot (-\sin(t)) + \frac{\cancel{2}z}{\cancel{2}(x^2+y^2+z^2)} \cdot \sec^2(t)$$

$\boxed{\begin{aligned} &x^2+y^2+z^2 \\ &= \sin^2(t)+\cos^2(t) \\ &+\tan^2(t) \\ &= 1+\tan^2(t) \\ &= \sec^2(t) \end{aligned}}$

$$\overset{\downarrow}{=} \frac{\sin(t)}{\sec^2(t)} \cos(t) - \frac{\cos(t)}{\sec^2(t)} \sin(t) + \frac{\tan(t)}{\sec^2(t)} \sec^2(t)$$

$$= \underline{\sec^2(t)}$$

8. Given $z = \arcsin(x-y)$, and $x = s^2+t^2$, $y = 1-2st$.

Then
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 2s + \frac{-1}{\sqrt{1-(x-y)^2}}\cdot(-2t)$$

$$\boxed{\begin{aligned}&(x-y)^2\\ &= (s^2+t^2-(1-2st))^2\\ &= ((s+t)^2-1)^2\end{aligned}}$$

$$= \frac{2s+2t}{\sqrt{1-((s+t)^2-1)^2}}$$ and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 2t + \frac{-1}{\sqrt{1-(x-y)^2}}(-2s) = \frac{2t+2s}{\sqrt{1-((s+t)^2-1)^2}}$$

14. Let $W(s,t) = F(u(s,t), v(s,t))$ where F, u, v are differentiable

Then $W_s(s,t) = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s}$ and $W_t(s,t) = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t}$

$= F_u\cdot u_s + F_v\cdot v_s$ $\qquad$ $= F_u u_t + F_v v_t$.

Thus $W_s(1,0) = F_u(u(1,0), v(1,0))\cdot u_s(1,0) + F_v(u(1,0), v(1,0))\, v_s(1,0)$

$$= F_u(2,3)\cdot u_s(1,0) + F_v(2,3)\cdot v_s(1,0)$$

$$= -1\cdot(-2) + 10\cdot 5 = \underline{52}.$$

and $W_t(1,0) = F_u(u(1,0), v(1,0))\, u_t(1,0) + F_v(u(1,0), v(1,0))\cdot v_t(1,0)$

$$= -1\cdot 6 + 10\cdot 4 = \underline{34}.$$

§14.5

22. Given $u=\sqrt{r^2+s^2}$, $r=y+x\cos(t)$, $s=x+y\sin(t)$ when $x=1, y=2, t=0$. Find out

$$\cdot\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}=\frac{2r}{2\sqrt{r^2+s^2}}\cdot\cos(t)+\frac{s}{\sqrt{r^2+s^2}}\cdot 1$$

Then $\left.\frac{\partial u}{\partial x}\right|_{(1,2,0)}=\frac{3}{\sqrt{3^2+1^2}}\cdot\cos(0)+\frac{1}{\sqrt{3^2+1^2}}=\frac{4}{\sqrt{10}}$.

as $(x,y,t)=(1,2,0)$
$r=2+1=3$
$s=1+0=1$

$$\cdot\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}=\frac{r}{\sqrt{r^2+s^2}}\cdot 1+\frac{s}{\sqrt{r^2+s^2}}\sin(t)$$

Then $\left.\frac{\partial u}{\partial y}\right|_{(1,2,0)}=\frac{3}{\sqrt{3^2+1^2}}\cdot 1+\frac{1}{\sqrt{3^2+1^2}}\cdot 0=\frac{3}{\sqrt{10}}$

$$\cdot\ \frac{\partial u}{\partial t}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial t}=\frac{r}{\sqrt{r^2+s^2}}\cdot x(-\sin(t))+\frac{s}{\sqrt{r^2+s^2}}\cdot y\cos(t)$$

Then $\left.\frac{\partial u}{\partial t}\right|_{(1,2,0)}=\frac{3}{\sqrt{3^2+1^2}}\cdot 1\cdot 0+\frac{1}{\sqrt{3^2+1^2}}\cdot 2\cdot 1=\frac{2}{\sqrt{10}}$.

48. Let $z=f(x,y)$, $x=s+t$, $y=s-t$. To show

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2\overset{?}{=}\frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t}$$

We have

$$\text{RHS}=\frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t}=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)\cdot\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$=\left(\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}\cdot 1\right)\left(\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}(-1)\right)=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\text{LHS}\ ▨$$

46. Let $u=f(x,y)$, $x=e^{s}\cos(t)$, $y=e^{s}\sin(t)$. To show

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 \overset{?}{=} e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right].$$

We have

$$RHS = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right] = e^{-2s}\left[\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)^2+\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)^2\right]$$

$$= e^{-2s}\left[\left(\frac{\partial u}{\partial x}\cdot e^{s}\cos(t)+\frac{\partial u}{\partial y}e^{s}\sin(t)\right)^2+\left(\frac{\partial u}{\partial x}e^{s}(-\sin(t))+\frac{\partial u}{\partial y}e^{s}\cos(t)\right)^2\right]$$

$$= e^{-2s}\left[e^{2s}\left(\left(\frac{\partial u}{\partial x}\right)^2\cos^2(t)+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos(t)\sin(t)+\left(\frac{\partial u}{\partial y}\right)^2\sin^2(t)\right)\right.$$

$$\left.+e^{2s}\left(\left(\frac{\partial u}{\partial x}\right)^2\sin^2(t)-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos(t)\sin(t)+\left(\frac{\partial u}{\partial y}\right)^2\cos^2(t)\right)\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2(t)+\sin^2(t)\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2(t)+\sin^2(t)\right)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = LHS$$

§14.6

4. Given $f(x,y)=x^2y^3-y^4$, point $(2,1)$ and angle $\theta=\frac{\pi}{4}$

Then the directional derivative of f at $(2,1)$ in the direction $\vec{u}=\langle\cos(\frac{\pi}{4}),\sin(\frac{\pi}{4})\rangle=\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$ is

$$D_{\vec{u}}f(x,y)\Big|_{(2,1)} = \nabla f(x,y)\cdot\vec{u}\Big|_{(2,1)} = \langle f_x, f_y\rangle\cdot\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle\Big|_{(2,1)}$$

$$= \langle 2xy^3, 3x^2y^2-4y^3\rangle\cdot\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle\Big|_{(2,1)} = 2\sqrt{2}+4\sqrt{2} = \underline{6\sqrt{2}}$$

6. Given $f(x,y) = x\sin(xy)$, point $(2,0)$ and angle $\theta = \frac{\pi}{3}$,
Then the directional derivative of f at $(2,0)$ in the direction $\vec{u} = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ is

$$D_{\vec{u}} f(x,y)\Big|_{(2,0)} = \nabla f(x,y) \cdot \vec{u}\Big|_{(2,0)} = \langle f_x, f_y \rangle \cdot \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle\Big|_{(2,0)}$$

$$= \langle \sin(xy) + x[\cos(xy)]y,\ x[\cos(xy)]x \rangle \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle\Big|_{(2,0)}$$

$$= \langle 0+0, 4 \rangle \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle = \underline{2\sqrt{3}}$$

8. Given $f(x,y) = \frac{y^2}{x}$, point $P(1,2)$ and a direction $\vec{u} = \langle \frac{2}{3}, \frac{\sqrt{5}}{3} \rangle$.
Then (a) gradient of f is $\nabla f(x,y) = \langle f_x, f_y \rangle = \underline{\langle \frac{-y^2}{x^2}, \frac{2y}{x} \rangle}$

(b) the gradient of f at P is $\nabla f(1,2) = \underline{\langle -4, 2 \rangle}$

(c) the rate of change of f at P in $\vec{u}$ is

$$D_{\vec{u}} f(x,y)\Big|_P = \nabla f(x,y) \cdot \vec{u}\Big|_P = \langle -4, 2 \rangle \cdot \langle \tfrac{2}{3}, \tfrac{\sqrt{5}}{3} \rangle$$

$$= \underline{\frac{-8+2\sqrt{5}}{3}}$$

$(|\vec{v}| = \sqrt{5})$

12. Given $f(x,y) = \ln(x^2+y^2)$, point $(2,1)$ and direction $\vec{v} = \langle -1, 2 \rangle$.
Then the directional derivative of f at $(2,1)$ in $\frac{\vec{v}}{|\vec{v}|}$ is

$$D_{\frac{\vec{v}}{|\vec{v}|}} f(x,y)\Big|_{(2,1)} = \nabla f(x,y) \cdot \frac{\vec{v}}{|\vec{v}|}\Big|_{(2,1)} = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle \cdot \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle\Big|_{(2,1)}$$

$$= \langle \frac{-4}{5}, \frac{1}{5} \rangle \cdot \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{4+2}{5\sqrt{5}} = \underline{\frac{6}{25}\sqrt{5}}$$

16. Given $f(x,y,z)=\sqrt{xyz}$, point $(3,2,6)$ and direction $\vec{v}=\langle -1,-2,2\rangle$

Then the directional derivative of f at $(3,2,6)$ in direction $\frac{\vec{v}}{|\vec{v}|}$

$(|\vec{v}|=\sqrt{1+4+4}=3)$

is $D_{\frac{\vec{v}}{|\vec{v}|}} f(x,y,z)\Big|_{(3,2,6)} = \nabla f(x,y,z)\cdot \frac{\vec{v}}{|\vec{v}|}\Big|_{(3,2,6)}$

$$= \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle \Big|_{(3,2,6)}$$

$$= \left\langle \frac{12}{12}, \frac{18}{12}, \frac{6}{12} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle = -\frac{1}{\sqrt{3}} + \frac{-3}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \underline{\frac{-3}{\sqrt{3}}}$$

31. The temperature T in a ball is inversely proportional to the distance from $(0,0,0)$

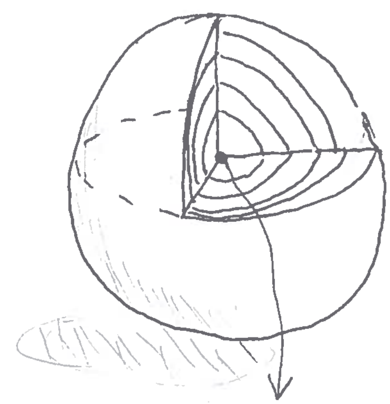
means :

(i) T at $(0,0,0)$ is the <u>Maximum</u>.

(ii) T is decreasing along the outward vector $\vec{r}$ from $(0,0,0)$.

such that $\underline{T(x,y,z)\cdot|\vec{r}|=K}$, where K is a constant.

Given $T=120°$ at point $(1,2,2)$.



(a) Find the rate of change of T at $(1,2,2)$ toward to point $(2,1,3)$

We have $\nabla_{\frac{\vec{v}}{|\vec{v}|}} T\Big|_{(1,2,2)}$ where $\vec{v}=(2,1,3)-(1,2,2)=\langle 1,-1,1\rangle$

$(|\vec{v}|=\sqrt{3})$

Then $\nabla_{\frac{\vec{v}}{|\vec{v}|}} T\Big|_{(1,2,2)} = \nabla T\cdot\frac{\vec{v}}{|\vec{v}|}\Big|_{(1,2,2)}$

Since $T(1,2,2)=120$. We have $\vec{r}=(1,2,2)-(0,0,0)$

$=\langle 1,2,2\rangle$

Then $k=T(1,2,2)\cdot|\vec{r}|=120\cdot\sqrt{1^2+2^2+2^2}=360$

$\Rightarrow$ For any point (x,y,z), we have $\vec{r}=\langle x,y,z\rangle$

and $T(x,y,z)\cdot|\vec{r}|=360$

$$\Rightarrow T(x,y,z)=\frac{360}{|\vec{r}|}=\frac{360}{\sqrt{x^2+y^2+z^2}}$$

Thus, $\nabla_{\frac{\vec{v}}{|\vec{v}|}}T\Big|_{(1,2,2)}=\nabla T\cdot\frac{\vec{V}}{|\vec{V}|}\Big|_{(1,2,2)}=\langle T_x,T_y,T_z\rangle\cdot\langle\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle\Big|_{(1,2,2)}$

$$=\langle-\frac{1}{\cancel{2}}\frac{360\cdot 2x}{(x^2+y^2+z^2)^{\frac{3}{2}}},-\frac{1}{\cancel{2}}\frac{360\cdot 2y}{(x^2+y^2+z^2)^{\frac{3}{2}}},-\frac{1}{\cancel{2}}\cdot\frac{360\cdot 2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\rangle\cdot\langle\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle\Big|_{(1,2,2)}$$

$$=\langle\frac{-360\cdot 1}{27},-\frac{360\cdot 2}{27},-\frac{360\cdot 2}{27}\rangle\cdot\langle\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle$$

$$=-\frac{360}{27}\left(+\frac{1}{\sqrt{3}}-\frac{2}{\sqrt{3}}+\frac{2}{\sqrt{3}}\right)=-\frac{40}{3\sqrt{3}}=-\frac{40}{9}\sqrt{3}.$$

(b) Since ∇T means the direction of fastest increase of T,

which is $\nabla T=\langle T_x,T_y,T_z\rangle=\langle-\frac{360x}{\sqrt{x^2+y^2+z^2}},-\frac{360y}{\sqrt{x^2+y^2+z^2}},-\frac{360z}{\sqrt{x^2+y^2+z^2}}\rangle$.

For any point (x_0,y_0,z_0), $\nabla T=\langle-\frac{360x_0}{\sqrt{x_0^2+y_0^2+z_0^2}},-\frac{360y_0}{\sqrt{x_0^2+y_0^2+z_0^2}},-\frac{360z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}\rangle$

is always a vector that has same direction of the vector $\langle-x_0,-y_0,-z_0\rangle$ with a scalar $\frac{-360}{\sqrt{x_0^2+y_0^2+z_0^2}}$ $\Rightarrow$ a vector from (x_0,y_0,z_0) toward to $(0,0,0)$.

37. Assume u, v are differentiable function w.r.t x and y and a, b are constant.

(a) $\nabla(a\vec{u}+b\vec{v}) = \langle (au+bv)_x, (au+bv)_y \rangle$

$= \langle a(u)_x + b(v)_x, a(u)_y + b(v)_y \rangle$

$= \langle a(u)_x, a(u)_y \rangle + \langle b(v)_x, b(v)_y \rangle$

$= a\langle u_x, u_y \rangle + b\langle v_x, v_y \rangle = a\nabla u + b\nabla v.$

(b) $\nabla(uv) = \langle (uv)_x, (uv)_y \rangle = \langle u_x v + u v_x, u_y v + u v_y \rangle$

$= \langle u_x v, u_y v \rangle + \langle u v_x + u v_y \rangle$

$= v\langle u_x, u_y \rangle + u\langle v_x, v_y \rangle = v\nabla u + u\nabla v.$

(c) $\nabla\left(\frac{u}{v}\right) = \left\langle \left(\frac{u}{v}\right)_x, \left(\frac{u}{v}\right)_y \right\rangle = \left\langle \frac{u_x v - v_x u}{v^2}, \frac{u_y v - v_y u}{v^2} \right\rangle$

$= \frac{1}{v^2}\left(\langle u_x v, u_y v \rangle - \langle v_x u, v_y u \rangle \right) = \frac{v\nabla u - u\nabla v}{v^2}$

(d) $\nabla u^n = \langle (u^n)_x, (u^n)_y \rangle = \langle n u^{n-1} \cdot u_x, n u^{n-1} u_y \rangle$

$= n u^{n-1} \langle u_x, u_y \rangle = n u^{n-1} \nabla u.$

(4) Given heat equation $u_t = \alpha^2 u_{xx}$ with Boundary Condition $u(0,t) = u(L,t) = 0$, To check $u(x,t) = e^{-\frac{\alpha^2\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right)$ is a solution of

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{—(1)} \\ u(0,t) = u(L,t) = 0 & \text{—(2)} \end{cases}$$

We have.

For (1),

$$u_t(x,t) = -\frac{\alpha^2\pi^2}{L^2} e^{-\frac{\alpha^2\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right) \quad \text{and}$$

$$u_x(x,t) = \frac{\pi}{L} e^{-\frac{\alpha^2\pi^2}{L^2}t} \cos\left(\frac{\pi x}{L}\right),$$

$$u_{xx}(x,t) = -\frac{\pi}{L} \cdot \frac{\pi}{L} e^{-\frac{\alpha^2\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right)$$

$$\Rightarrow \alpha^2 u_{xx}(x,t) = -\alpha^2 \frac{\pi^2}{L^2} e^{-\frac{\alpha^2\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right) = u_t(x,t).$$

For (2).

$$u(0,t) = e^{-\frac{\alpha^2\pi^2}{L^2}t} \underbrace{\sin\left(\frac{\pi \cdot 0}{L}\right)}_{\sin(0)=0} = 0.$$

$$\cancel{\lim_{L\to 0} \left| \frac{\sin\left(\frac{\pi x}{L}\right)}{e^{\frac{\alpha^2\pi^2 t}{L^2}}} \right| \leq \lim_{L\to 0} \left| \frac{1}{e^{\frac{\alpha^2\pi^2 t}{L^2}}} \right| = 0}$$

and

$$u(L,t) = e^{-\frac{\alpha^2\pi^2}{L^2}t} \underbrace{\sin\left(\frac{\pi \cdot L}{L}\right)}_{\sin(\pi)=0} = 0$$

∎