Honors Calculus, Math 1451, HW3, Solution

§13.4

16. Given $\vec{a}(t) = 2\vec{i} + 6t\vec{j} + 12t^2\vec{k}$, and $\vec{v}(0) = \vec{i}$, $\vec{r}(0) = \vec{j} - \vec{k}$.

Find the velocity $\vec{v}(t)$, we have

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du$$

$$= \langle 1,0,0\rangle + \int_0^t [2\vec{i} + 6u\vec{j} + 12u^2\vec{k}]\, du$$

$$= \langle 1,0,0\rangle + (2t\vec{i} + 3t^2\vec{j} + 4t^3\vec{k}) = \langle 2t+1, 3t^2, 4t^3\rangle$$

and for the position $\vec{r}(t)$, we have

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du = \langle 0,1,-1\rangle + \int_0^t [(2u+1)\vec{i} + 3u^2\vec{j} + 4u^3\vec{k}]\, du$$

$$= \langle 0,1,-1\rangle + (t^2+t)\vec{i} + (t^3)\vec{j} + (t^4)\vec{k} = \langle t^2+t, t^3+1, t^4-1\rangle$$

18. (a) Given $\vec{a}(t) = t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$, $\vec{v}(0) = \vec{k}$, $\vec{r}(0) = \vec{j} + \vec{k}$.

The velocity $\vec{v}(t)$ which has the given acceleration and initial velocity is $\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du = \vec{k} + \int_0^t [u\vec{i} + e^u\vec{j} + e^{-u}\vec{k}]\, du$

$$= \vec{k} + \left[\frac{t^2}{2}\vec{i} + e^t\vec{j} - e^{-t}\vec{k}\right] = \frac{t^2}{2}\vec{i} + e^t\vec{j} + (-e^{-t}+1)\vec{k}$$ and

The position vector which has ~~the~~ given initial position is

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du = \vec{j} + \vec{k} + \int_0^t \left[\frac{u^2}{2}\vec{i} + e^u\vec{j} + (-e^{-u}+1)\vec{k}\right] du$$

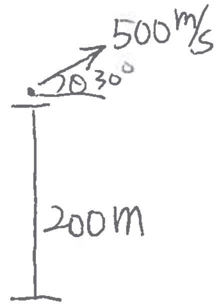
$$= \vec{j}+\vec{k}+\frac{t^3}{6}\vec{i}+e^{t}\vec{j}+(e^{-t}+t)\vec{k}.$$

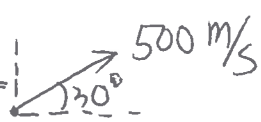
$$= \frac{t^3}{6}\vec{i}+(e^{t}+1)\vec{j}+(e^{-t}+t+1)\vec{k}$$

(b).

24. A projectile is fired from a position 200m above the ground with an initial speed 500 m/s and angle of elevation 30°.

, We have



$\vec{V}(0)=$ 

$$= 500\cdot\cos(30^\circ)\vec{i} + 500\cdot\sin(30^\circ)\vec{j}$$

$$= 250\sqrt{3}\,\vec{i}+250\vec{j}$$

and $\vec{r}(0)=200\vec{j}$. and $\vec{a}(t)=-9.8\,\vec{j}$

Then $\vec{v}(t)=\vec{v}(0)+\int_0^t \vec{a}(u)du = (250\sqrt{3}\,\vec{i}+250\vec{j})+(-9.8t)\vec{j}$

$$= 250\sqrt{3}\,\vec{i}+(250-9.8t)\vec{j}.$$

and $\vec{r}(t)=\vec{r}(0)+\int_0^t \vec{v}(u)du = 200\vec{j}+(250\sqrt{3}t)\vec{i}+(250t-4.9t^2)\vec{j}$

$$=(250\sqrt{3}t)\vec{i}+(200+250t-4.9t^2)\vec{j}.$$

(a) The range of projectile is to find the farest $\vec{i}$-direction of the ball. Let $y(t)=200+250t-4.9t^2$,

$$y(t)=0 \Rightarrow t=\frac{-250\pm\sqrt{58580}}{-9.8}=\frac{250\pm\sqrt{58580}}{9.8}$$

So as $t = \frac{250+\sqrt{58580}}{9.8}$, the ball hits the ground

and $x(t) = 250\sqrt{3}t = 250\sqrt{3}\left(\frac{250+\sqrt{58580}}{9.8}\right)$ is the farest distance.

(b) the maximum height reached is finding the t as $y'(t)=0$

$\Rightarrow y'(t) = 250 - 9.8t = 0 \Rightarrow t = \frac{250}{9.8}$

Then $y(t) = 200 + 250 \cdot \frac{250}{9.8} - 4.9\left(\frac{250}{9.8}\right)^2$ is the maximum height of the ball.

(c) the speed at impact is as the ball hits the ground.

as $t = \frac{250+\sqrt{58580}}{9.8}$, we have

$\vec{V}(t) = 250\sqrt{3}\vec{i} + \left(250 - 9.8 \cdot \frac{250+\sqrt{58580}}{9.8}\right)\vec{j}$

$= 250\sqrt{3}\vec{i} + (-\sqrt{58580})\vec{j}$.

40. Given a particle with mess m with position vector $\vec{r}(t)$,

Then angular momentum is $\vec{L}(t) = m\vec{r}(t) \times \vec{v}(t)$ and its torgue is $\vec{\tau}(t) = m\vec{r}(t) \times \vec{a}(t)$, we have

$$\vec{L}'(t) = m\vec{r}'(t) \times \vec{v}(t) + m\vec{r}(t) \times \vec{v}'(t)$$
$$= m\vec{v}(t) \times \vec{v}(t) + m\vec{r}(t) \times \vec{a}(t)$$
$$= 0 + m\vec{r}(t) \times \vec{a}(t) = \vec{\tau}(t)$$

$(\vec{v} \times \vec{v} = |\vec{v}||\vec{v}| \cdot \sin(0) = 0)$

So if $\vec{\tau}(t)=\vec{0}$ for all t, we have

$\vec{L}'(t)=\vec{0} \Rightarrow \vec{L}(t)$ is constant.

42. Given the velocity of a rocket $\vec{v}(t)$, mass $m(t)$, the velocity of the escape gases $\vec{v}_e$, and we have

$$m\frac{d\vec{v}}{dt}=\frac{dm}{dt}\cdot\vec{v}_e \Rightarrow \frac{d\vec{v}}{dt}=\frac{1}{m(t)}\frac{dm(t)}{dt}\vec{v}_e$$

$$=\frac{d[\ln(m(t))]}{dt}\cdot\vec{v}_e$$

(a). Then we have $\vec{v}(t)$

$$\vec{v}(t)=\vec{v}(0)+\int_0^t \vec{v}'(u)du=\vec{v}(0)+[\ln(m(u))]\vec{v}_e\Big|_0^t$$

$$=\vec{v}(0)+\{\ln[m(t)]-\ln[m(0)]\}\vec{v}_e$$

$$=\vec{v}(0)-\left\{\ln\left[\frac{m(0)}{m(t)}\right]\right\}\cdot\vec{v}_e$$

(b) $\vec{v}(0)=0 \rightarrow \vec{v}(t)=-2\vec{v}_e$

First, find the t such that $\vec{v}(t)=2\vec{v}_e$.

$$\Rightarrow -2\vec{v}_e=\vec{v}(t)=\vec{v}(0)-\left\{\ln\left[\frac{m(0)}{m(t)}\right]\right\}\vec{v}_e=0-\left\{\ln\left[\frac{m(0)}{m(t)}\right]\right\}\vec{v}_e$$

$$\Rightarrow -2=-\ln\left[\frac{m(0)}{m(t)}\right] \Rightarrow \frac{m(0)}{m(t)}=e^{+2} \Rightarrow \frac{m(t)}{m(0)}=e^{-2}$$

$\Rightarrow$ The rocket have to burn $1-e^{-2}$ of its own initial mass.

Homework 3

Math 1451 Accelerated Calculus Spring 2016

Problem 1. Showing Kepler's 2nd Law
Solution 1.
Given $e_r = (cos\theta, sin\theta)$ and $e_\theta = (-sin\theta, cos\theta)$ we have:

$$r = re_r \text{ and } \dot{e_r} = \dot{\theta}(-sin\theta, cos\theta) = \dot{\theta}e_\theta.$$

Our goal is to find acceleration.

$$v = \frac{dr}{dt} = \dot{r}e_r + r\dot{e_r} = \dot{r}e_r + r\dot{\theta}e_\theta$$
$$a = \frac{dv}{dt} = \ddot{r}e_r + \dot{r}\dot{e_r} + (\dot{r}\dot{\theta} + r\ddot{\theta})e_\theta + r\dot{\theta}\dot{e_\theta}$$

Now,

$$\dot{e_\theta} = \dot{\theta}(-cos\theta, -sin\theta) = -\dot{\theta}e_r$$

Rewriting acceleration gives:

$$a = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta = fe_r$$

We are looking good now! Since e_r and e_θ are orthogonal we can break acceleration into 2 parts:

$$\ddot{r} - r\dot{\theta}^2 = f \text{ and}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \text{ since all force is in the radial direction.}$$

We conclude

$$\frac{d(r^2\dot{\theta})}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Solution 2.
Given that angular momentum is conserved we have:

$$C = r \times \dot{r} = (rcos\theta i + rsin\theta j) \times (\dot{r}cos\theta - rsin\theta\dot{\theta})i + (\dot{r}sin\theta + rcos\theta)j$$

i	j	k
$rcos\theta$	$rsin\theta$	0
$\dot{r}cos\theta - rsin\theta$	$\dot{r}sin\theta + rcos\theta\dot{\theta}$	0

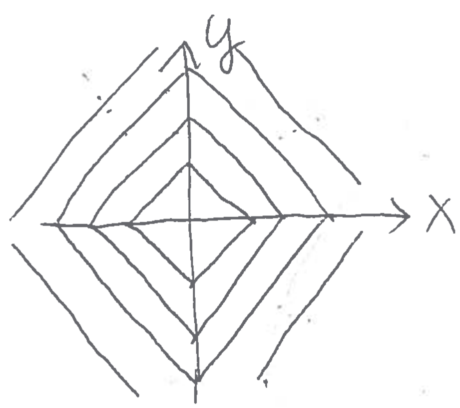
$$= (r\dot{r}cos\theta sin\theta + r^2cos^2\theta\dot{\theta} - r\dot{r}cos\theta sin\theta + r^2sin^2\theta)k$$
$$= r\dot{r}(e_r \times e_r) + r(r\dot{\theta})k = r(r\dot{\theta})k$$

(2)

Please see the answers from Meagan (front page)

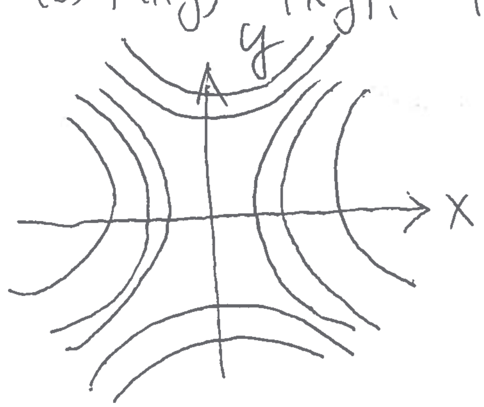
§14.1

30. (a) $f(x,y) = |x| + |y|$. The contour map of $f(x,y)$ is



$\Rightarrow$ VI

(b) $f(x,y) = |xy|$. The contour map of $f(x,y)$ is

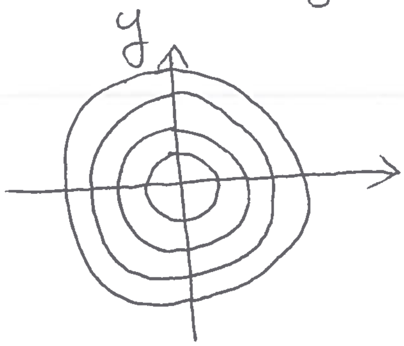


$\Rightarrow$ V

DE

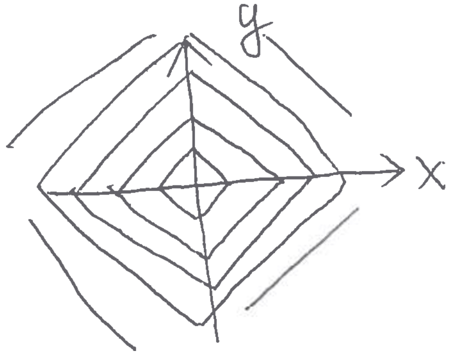
30

(c) $f(x,y)=\dfrac{1}{1+x^2+y^2}$, its contour map looks like



$\Rightarrow$ I

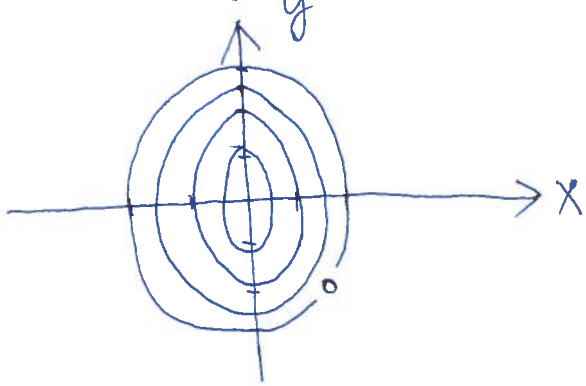
(f) $f(x,y)=\sin(|x|+|y|)$, its contour map looks like



and sine function is periodic $\Rightarrow$ III

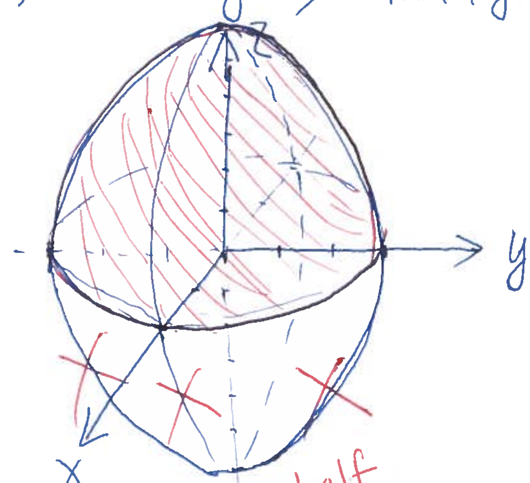
48. $f(x,y)=\sqrt{36-9x^2-4y^2}$

If $f(x,y)=k$, we have $k^2=36-9x^2-4y^2 \Rightarrow 9x^2+4y^2=36-k^2$



contour map

$\Rightarrow$



graph: It is ~~an~~ a half ellipsoid

(x-radius $=2$, y-radius $=3$, z-radius $=6$)

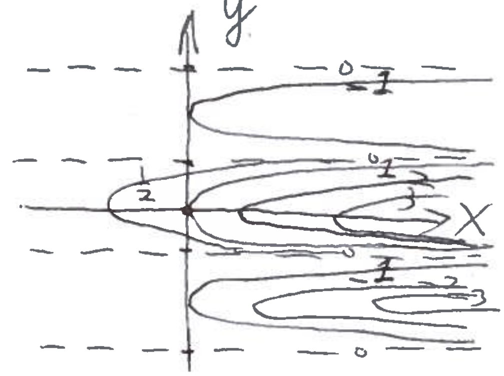
§14.1

56. $z=e^x\cos(y)$

(a) graph

by (b) $\Rightarrow$ (A)

(b) contour map



$\Rightarrow$ IV

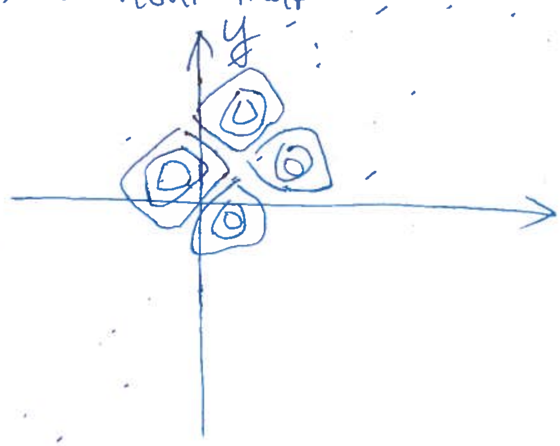
as $y=\frac{(2n-1)}{2}\pi \Rightarrow z=\begin{cases} e^x & , n=1,3,5 \\ -e^x & n=2,4,6 \end{cases}$

58. $z=\sin(x)-\sin(y)$

(a) graph

by (b) $\Rightarrow$ (E)

(b) contour map



$\Rightarrow$ II

Both x, y direction are periodic

So when one is fixed, the other will be a sine function

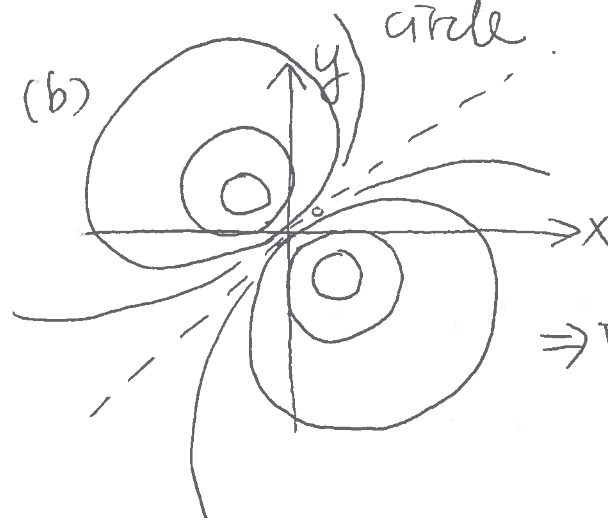
60. $z = \dfrac{x-y}{1+x^2+y^2}$

let $z=k$. $\Rightarrow k(1+x^2+y^2)=x-y \Rightarrow kx^2+ky^2+k-x+y=0$

$\Rightarrow k(x^2-\frac{x}{k}+(\frac{1}{2k})^2)+k(y^2+\frac{y}{k}+(\frac{1}{2k})^2)+k-\frac{1}{2k}=0$

$\Rightarrow k(x-\frac{1}{2k})^2+k(y+\frac{1}{2k})^2+\frac{2k^2-1}{2k}=0$ which is circle.

(b)



$\Rightarrow$ V

(a) by (b) $\Rightarrow$ (D)

§14.2

6. $\lim\limits_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$

let $x=r\cos\theta-1$, $y=r\sin\theta+1$, we have.

$\lim\limits_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y) = \lim\limits_{r\to0} e^{-(r\cos\theta-1)(r\sin\theta+1)}\cos(r\cos\theta+r\sin\theta)$

$= \lim\limits_{r\to0} e^{-[r^2\cos\theta\sin\theta-r\sin\theta+r\cos\theta-1]}\cos(r\cos\theta+r\sin\theta)$

$= e^1\cos(0) = e^1$. limit exists and equals e^1.

8. $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$

Let $x = r\cos\theta + 1$, $y = r\sin\theta$, we have

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \lim_{r\to 0} \ln\left(\frac{1+r^2\sin\theta}{r^2\cos\theta - 2r\cos\theta + 1 + r^2\cos\theta\sin\theta}\right) = \ln(1) = 0$$

Limit exists and equal 0.

12. $\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$

① Check path $x=y$. we have $\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4} = \lim_{x\to 0} \frac{6x^4}{3x^4} = 2$.

② Check path $x=-y$, we have $\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4} = \lim_{x\to 0} \frac{-6x^4}{3x^4} = -2$

Since we got two different limits for two different paths. The Limit doesn't exist.

30. $F(x,y) = \frac{x-y}{1+x^2+y^2}$

First, the domain of $F(x,y)$ is $\mathbb{R}^2$ or $\{(x,y) \mid -\infty < x, y < \infty\}$ and $\lim_{(x,y)\to(a,b)} F(x,y) = \frac{a-b}{1+a^2+b^2}$, Then the set of points $f(x,y)$ is continuous is $\mathbb{R}^2$.

36. $f(x,y,z)=\sqrt{x+y+z}$

First, if $x+y+z\geq 0$. $f(x,y,z)$ is defined.

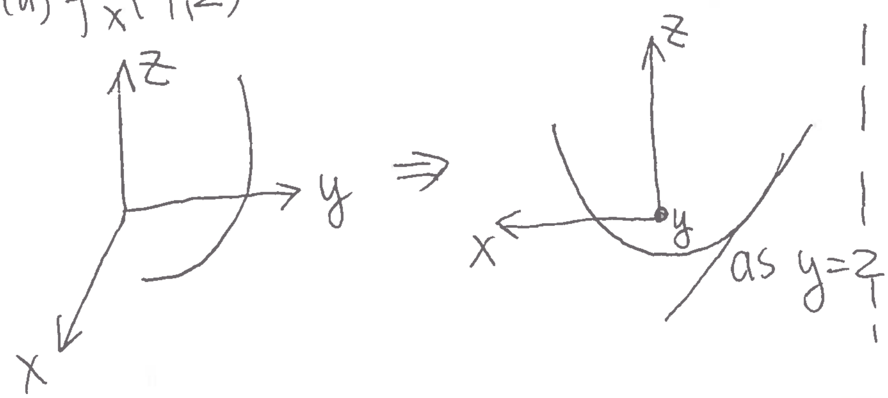
Then since $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z)=\sqrt{a+b+c}$ if $a+b+c\geq 0$.

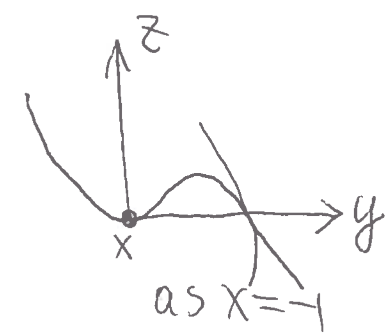
Thus the set of point $f(x,y,z)$ is continuoues is

$\{(x,y,z)\mid x+y+z\geqslant 0\}$.

§14.3

6. (a) $f_x(-1,2)<0$ (b) $f_y(-1,2)<0$.

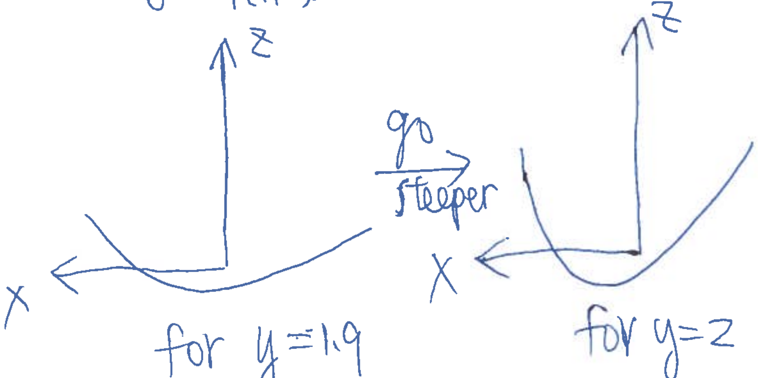




8. (a) $f_{xy}(1,2)>0$. $f_{xy}(-1,2)<0$

$\| \qquad\qquad\qquad\qquad \|$

$\frac{\partial}{\partial y}(f_x)\Big|_{(1,2)}$ ⇐ the rate of change of f_x on y-direction ⇒ $\frac{\partial}{\partial y}(f_x)\Big|_{(-1,2)}$



16. Given $f(x,y) = x^4y^3 + 8x^2y$.

Then $f_x(x,y) = 4x^3y^3 + 16xy$ and $f_y(x,y) = 3x^4y^2 + 8x^2$.

18. Given $f(x,t) = \sqrt{x}\ln(t)$.

Then $f_x(x,t) = \frac{1}{2\sqrt{x}}\ln(t)$ and $f_t(x,t) = \frac{\sqrt{x}}{t}$

20. Given $z = \tan(xy)$.

Then $\frac{\partial z}{\partial x} = y \cdot \sec^2(xy)$ and $\frac{\partial z}{\partial y} = x\sec^2(xy)$

28. Given $f(x,y) = \int_y^x \cos(t^2)\,dt = \int_0^x \cos(t^2)\,dt - \int_0^y \cos(t^2)\,dt$

Then $f_x(x,y) \underset{\substack{\uparrow \\ \text{F.T.C}}}{=} \cos(x^2)$ and $f_y(x,y) = -\cos(y^2)$.

37. Given $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Then $\frac{\partial u}{\partial x_1} = \frac{2x_1}{2\sqrt{x_1^2+x_2^2+\cdots+x_n^2}} = \frac{x_1}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$,

$\frac{\partial u}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$, $\cdots$, $\frac{\partial u}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$, $\cdots$,

$\frac{\partial u}{\partial x_n} = \frac{x_n}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$

38. Given $u = \sin(x_1 + 2x_2 + \cdots + n x_n)$

Then $\frac{\partial u}{\partial x_1} = 1 \cdot \cos(x_1 + 2x_2 + \cdots + n x_n)$, $\frac{\partial u}{\partial x_2} = 2\cos(x_1 + 2x_2 + \cdots + n x_n)$, ……

$$\frac{\partial u}{\partial x_j} = j \cos(x_1 + 2x_2 + \cdots + j x_j + \cdots + n x_n), \ \cdots$$

$$\frac{\partial u}{\partial x_n} = n \cos(x_1 + 2x_2 + \cdots + j x_j + \cdots + n x_n).$$

46. Given $yz = \ln(x+z)$

To Find $\frac{\partial z}{\partial x}$, we do "$\frac{\partial}{\partial x}$" on both sides and get

$$y \frac{\partial z}{\partial x} = \frac{1}{x+z} \cdot \left(1 + \frac{\partial z}{\partial x}\right) \Rightarrow \left(y - \frac{1}{x+z}\right)\frac{\partial z}{\partial x} = \frac{1}{x+z}.$$

$$\Rightarrow \frac{\partial z}{\partial x} = y(x+z) - 1$$

Similary, doing "$\frac{\partial}{\partial y}$" on both sides and we get

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z} \cdot \frac{\partial z}{\partial y} \Rightarrow \left(\frac{1}{x+z} - y\right)\frac{\partial z}{\partial y} = z \Rightarrow \frac{\partial z}{\partial y} = \frac{z(x+z)}{x+z-y}.$$

48. Given $\sin(xyz) = x + 2y + 3z$.

Doing "$\frac{\partial}{\partial x}$" on both sides and we get $\cos(xyz) \cdot \left[yz + xy\frac{\partial z}{\partial x}\right] = 1 + 3\frac{\partial z}{\partial x}$

$$\Rightarrow \left(xy\cos(xyz) - 3\right)\frac{\partial z}{\partial x} = 1 - yz\cos(xyz) \Rightarrow \frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}.$$

Doing "$\frac{\partial}{\partial y}$" on both sides and we get $\cos(xyz) \cdot \left[xz + xy\frac{\partial z}{\partial y}\right] = 2 + 3\frac{\partial z}{\partial y}$.

$$\Rightarrow \left(xy\cos(xyz) - 3\right)\frac{\partial z}{\partial y} = 2 - xz\cos(xyz) \Rightarrow \frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}$$