Honors Calculus, Math 1451, Exam 1. sample2. Solution

(1) Given a plane $2x+4y-z=5$ and a line $x=4+3t$, $y=3$, $z=-2t$.

To find the intersection point of given plane, we have

$$2(4+3t)+4\cdot 3-(-2t)=5 \Rightarrow 8t=-15 \Rightarrow t=\frac{-15}{8}$$

$$\Rightarrow \text{point } \left(\frac{13}{8}, 3, \frac{15}{4}\right)$$

(2) Given: two vectors $\langle 1,-2,0\rangle$ and $\langle 1,0,3\rangle$, Then

$$\vec{n}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -6\vec{i}-3\vec{j}+2\vec{k}$$ is a vector orthogonal

to both of the given vectors and the unit vector of $\vec{n}$

is $$\frac{\vec{n}}{|\vec{n}|}=\frac{-6\vec{i}-3\vec{j}+2\vec{k}}{\sqrt{49}}=-\frac{6}{7}\vec{i}-\frac{3}{7}\vec{j}+\frac{2}{7}\vec{k}.$$

(3) Given a surface $z=f(x,y)=e^x\cos(x+y)$ and a point $(0,\frac{\pi}{3})$.

To find the tangent plane of f at $(0,\frac{\pi}{3})$, we have

$z=f(0,\frac{\pi}{3})=e^0\cos(\frac{\pi}{3})=\frac{1}{2}$ and $f_x(x,y)=e^x\cos(x+y)-e^x\sin(x+y)$

$f_y(x,y)=-e^x\sin(x+y)$

then the tangent plane is

$$z-\frac{1}{2}=f_x(0,\tfrac{\pi}{3})(x-0)+f_y(0,\tfrac{\pi}{3})(y-\tfrac{\pi}{3})$$

$$\Rightarrow z-\frac{1}{2}=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)(x-0)+\left(-\frac{\sqrt{3}}{2}\right)\left(y-\frac{\pi}{3}\right).$$

and the linearization of f at $(0, \frac{\pi}{3})$ is

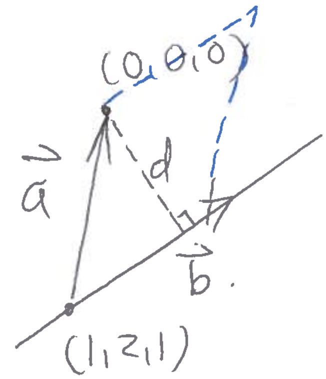
$$f(x,y) \approx \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)(x-0) - \frac{\sqrt{3}}{2}\left(y - \frac{\pi}{3}\right)$$

so $$f\left(\frac{1}{10}, \frac{\pi}{2}\right) \approx \left(\frac{1-\sqrt{3}}{2}\right)\frac{1}{10} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1-\sqrt{3}}{20} - \frac{\sqrt{3}}{12}\pi$$

(4) Given a parametric equation of line:

$$x = 1+2t,\ y = 2-t,\ z = t+1.$$

To Find the distance from origin to the given line, we pick up a point on line: $(1,2,1)$ and



let $\vec{a} = (0,0,0) - (1,2,1) = \langle -1,-2,-1 \rangle$ and $\vec{b} = \langle 2,-1,1 \rangle$ be the direction of the line. Then the distance

$$d = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} = \frac{\left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -1 \\ 2 & -1 & 1 \end{matrix} \right|}{\sqrt{4+1+1}} = \frac{|-3\vec{i} - \vec{j} + 5\vec{k}|}{\sqrt{6}} = \sqrt{\frac{35}{6}}.$$

(5) Given equation $P(V,T) = R\frac{T}{V}$. To Find linearization of P at $(5,400)$ and find the estimation of $P(5.05, 402)$, we have

$$\begin{pmatrix} P_V(V,T) = -R\frac{T}{V^2} \\ P_T(V,T) = \frac{R}{V} \end{pmatrix}$$

$$P(V,T) \approx P_V(\overset{5}{\overset{\shortparallel}{V}}, \overset{400}{\overset{\shortparallel}{T}})(V-5) + P_T(\overset{5}{\overset{\shortparallel}{V}}, \overset{400}{\overset{\shortparallel}{T}})(T-400) + P(5,400)$$

$$= -R\frac{400}{25}(V-5) + \frac{R}{5}(T-400) + 80R.$$

Then $P(5.05, 402) = -R\frac{400}{25}(0.05) + \frac{R}{5}(2) = -\frac{2}{5}R + 80R$

Thus $P(5.05, 402) - P(5,400) = -\frac{2}{5}R$

(6) Given wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$.

To show $u(x,t) = \sin(kx)\sin(wt)$ where $k = \frac{w}{v}$, we have

$\frac{\partial u}{\partial x} = k\cos(kx)\sin(wt)$ and $\frac{\partial u}{\partial t} = [\sin(kx)] \cdot w \cdot \cos(wt)$

$\frac{\partial^2 u}{\partial x^2} = -k^2\sin(kx)\sin(wt)$ and $\frac{\partial^2 u}{\partial t^2} = -[\sin(kx)]\, w^2 \cdot \sin(wt)$

Then

$$\frac{\partial^2 u}{\partial x^2} = -k^2\sin(kx)\sin(wt) \overset{k=\frac{w}{v}}{=} \frac{-w^2}{v^2}\sin(kx)\sin(wt) = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

(7) Given $z = f(x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial r}}_{\cos\theta} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta \quad \text{and}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta\right) + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\sin\theta\right)$$

$+ \frac{\partial f}{\partial y}\frac{\partial}{\partial r}(\sin\theta) = 0$

$+ \frac{\partial f}{\partial x}\frac{\partial}{\partial r}(\cos\theta) = 0$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\theta\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\theta\right)\frac{\partial y}{\partial r} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\sin\theta\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\sin\theta\right)\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y\partial x}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$